Liquidity Needs in Economies with Interconnected Financial Obligations

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Abstract

A model is developed where firms in a financial system have to settle their debts to each other by using a liquid asset (or money). The question that is studied is how many firms must have access to this asset from outside the financial system to make sure that all debts within the system are settled. The main result is that these liquidity needs are larger when these firms are more interconnected through their debts, *i.e.*, when they borrow from and lend to more firms. Two pecuniary externalities are discussed. One is the result of paying one creditor first rather than another. The second occurs when firms increase their financial transactions and thereby make it more likely that others will default. Lastly, the paper shows that interconnections can raise the number of firms that must be endowed with liquidity even when payments paths are chosen by a planner that seeks to avoid defaults. (JEL G20, D53, D85)

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The last few years have seen an explosion of two types of financial transactions. First, the volume of derivative instruments that are purchased and sold has ballooned. To give just one example, the face value of "credit derivative swaps" may have reached $60 trillion by May 2008.[1] Many of these derivative contracts require one side or another to make payments at pre-specified points in time. These required payments fluctuate in value and, to hedge against the resulting risks, many participants in these markets engage in simultaneous transactions with several parties so that their net payments are typically small. The second, and related, change is that a vast number of new intermediaries have been created whose main activity consists in engaging in financial market transactions. In addition to being active in derivatives markets these intermediaries borrow from other financial firms while simultaneously acquiring claims on others.

This raises the obvious question of whether this interconnectedness strengthens or weakens the financial system as a whole. Within this broad question, the current paper focuses on a narrow one. It focuses on a situation where every firm is solvent in the sense that the payments that any particular firm is expected to make do not exceed the payments it is entitled to receive. It then asks whether interconnectedness exacerbates the difficulties that firms have in meeting their obligations in periods where liquidity is more difficult to obtain.

In practice, firms make their payments with money, and legal tender laws ensure that this medium is always acceptable. A difference between financial firms and other economic actors is that, at least in "normal" times, these firms are continuously involved in markets such as the repo market where assets are exchanged for money within the day. Outside of liquidity crises, the range of assets that can be used for this purpose is relatively large. Solvent firms thus have no difficulty making their required payments. The cost involved in temporarily borrowing funds to make payments and returning these funds as soon as payments are received is small enough that it can be neglected relative to the difficulties encountered in liquidity crises. In such crises, short term funding becomes more difficult, as manifest for example in the reductions in the volume of "repo" transactions after the collapse

[1] See Reguly (2008) who also discusses the relationship of this volume to hedge funds.

of Lehman Brothers in late 2008.[2] One simple way of thinking about this is that firms have a collection of assets that they can convert into money immediately, and that this collection shrinks during liquidity crises. Yet another, and somewhat more abstract rendition of this is that firms are able to make payments by issuing IOU's in normal times, while they need access to particular liquid assets in periods of crisis.

This paper does not tackle the difficult problem of deriving such liquidity crises from more primitive assumptions. Rather, it takes at face value that firms sometimes have difficulty borrowing in short-term markets even when the model itself requires that such difficulties be "irrational." Perhaps the easiest way of thinking about this is to suppose that, at certain rare and unusual moments, potential lenders are subject to the irrational belief that these firms will not repay any loans that they are granted. A somewhat more conventional explanation is that potential lenders become extremely risk averse and that they believe that there is a small probability that these firms are insolvent. The current paper can then be thought of as applying in those states of nature where all firms happen to be solvent.

In addition to the assets that remain convertible into money, financial firms have two other sources of money during crises. First, at least some of them receive funds from debtors outside the financial system. Second, they have the funds that they receive from other financial firms that owe them funds in turn. Indeed, the ability of firms to use funds they receive from their debtors to pay off their creditors implies that a dollar of "exogenous" liquid assets in the financial system can be used to settle more than a single one dollar obligation. Nonetheless, I show that the interconnectedness of the financial system impairs the system's endogenous capacity to use these exogenous liquid assets multiple times. As a result, more interconnected financial systems require more liquidity (from exogenous sources) to settle a given volume of debt.[3]

[2] See Michael Mackenzie, "Awaiting the return of the repo market," *Financial times*, December 17, 2008. One reason for this reduction in volume may have been that hedge funds became more reluctant to let brokers borrow in repo markets by using securities belonging to hedge funds as collateral. This reduced brokers' access to liquid securities. See James Mackintosh, "Funds seek safer place to stash assets," *Financial Times*, January 12 2009.

[3] While I have emphasized that interconnection matter in the event of a liquidity crisis, a quite different interpretation is possible. In this alternate interpretation, what varies is the range of assets that is regarded

This result arises for two distinct reasons. The first is that the partial payments of debts generate a pecuniary externality when firms have multiple creditors. A firm whose liquid assets are positive but insufficient to pay all his creditors has to choose which creditor to pay off. The creditors (and the creditors' creditors) care about this choice even though the nature of actual contracts leads their influence on this choice to be negligible.

From the point of view of the financial system as a whole, the choice of whom to pay off generally matters as well. Imagine, in particular, that one firm has no further outstanding obligations, perhaps because it started out with sufficient liquid assets to pay off all his creditors. Then, another firm's decision as to whether it should pay off this firm or pay off another that also has further obligations affects the total volume of debts that is extinguished with the given supply of liquidity.

As the financial system becomes more interconnected, debtors with limited funds face a larger array of potential recipients for these funds. It then becomes easier to envisage situations where these funds go to firms that either have no further obligations themselves, or that have creditors with no further obligations. There thus exist "worst case scenarios" where the existing liquidity settles many fewer obligations than is theoretically possible. By way of contrast, this problem does not arise when each firm has only one creditor. Firms then have no choice regarding whom they pay and this reduces the scope for "wasting" payments on firms that have no further obligations left. This lends credence to the idea that the difficulties caused by periods of scarce liquidity are exacerbated when there exists a larger set of debt connections among firms.

Because I am concerned with the capacity of solvent firms to settle their obligations in situations where interconnectedness differs, the paper focuses on a system where all firms have current claims that equal or exceed their current obligations. This paper is thus complementary to the literature discussing the role of interconnectedness when some firms have

as acceptable as a means of payment. As emphasized by Kiyotaki and Wright (1993) and Wright (1997), extrinsic beliefs on whether an asset is acceptable can influence whether an asset is in fact acceptable for this purpose. Thus the paper can be thought of as analyzing the effect of interconnections on what happens when fewer assets are accepted as payment devices.

total current liabilities that exceed their current assets. This literature includes the studies by Allen and Gale (2000), Freixas *et al.* (2000), Eisenberg and Noe (2001), Cifuentes *et al* (2005) and Nier *et al* (2007).[4]

Like Eisenberg and Noe (2001), Cifuentes *et al* (2005) and Nier *et al* (2007), the current paper uses graph-theoretic techniques. One difference is that I use numerical techniques sparingly, with most of the results being established analytically. The cost of this, of course, is that I am able to do this only for relatively simple environments.

My focus on solvent institutions that are subject to trading frictions leads the model to be closely related to the literature that analyzes interbank payment systems. In such systems, banks send messages telling one another that they wish to make a payment. In "real time gross settlements" (RTGS) systems, this message is supposed to lead to an immediate debit to the paying bank (and a credit to the receiving bank). If the paying bank lacks funds and does not receive a loan, these debits and credits are not possible in a pure RTGS system, and one solution is to put them on hold. This was the solution adopted by the Swiss SIC system in the period 1987–1999, when it offered no loans to banks (see Martin 2005).

This solution seems inefficient when, using the terminology of Bech and Soramäki (2001), there is "gridlock" in that bank A lacks X dollars that it wishes to pay to bank B, who lacks X dollars that it wishes to pay to bank C, who in turn lacks X dollars that it wishes to pay to bank A. In such cases, it is more efficient to "net" the positions of these three banks. Some settlements systems, such as CHIPS, are designed to look for sets of payment messages that can be netted. These systems clear these sets of payments as soon as they

[4] Current obligations can exceed expected receipts from assets either because the firm is insolvent long term or because, as in the Diamond and Dybvig (1983) model, contracts are written in such a way that firms can only meet their short term commitments if a subset of the agents who are entitled to withdraw funds do so. This latter situation is often described as one of illiquidity, and it is useful to note the similarities and differences between this notion of illiquidity and the problems of liquidity faced by the firms in my model. What is similar is that, in both cases, firms have difficulty converting their existing claims into assets that can be used to pay their current obligations. The difference is that, in the Diamond and Dybvig (1983) setup, it is physically impossible to pay all claim holders in the short run the contracted-upon value of their claim. This makes good sense if one thinks of claim-holders as net lenders who, indirectly, are asking the net borrowers to convert their physical capital into consumer goods. Here, by contrast, all firms involved are in the financial sector, so there is no need to convert physical assets from one use to another to accomplish this.

are found. A common alternative, used both by the Fedwire (the U.S. Federal Reserve's settlements system) and the Swiss system after 1999, is to simply offer loans ("daylight overdrafts") to banks that lack sufficient funds to complete their desired payments.[5] The study of settlements systems thus shows that "netting" and the provision of government liquidity can be substitutes for dealing with gridlock.

The current study is related to this literature because it considers situations where, since all firms have claims that are at least as large in value as their obligations, the financial system would operate smoothly if there were extensive netting. My analysis, however, is more applicable to firms like investment banks and other "nonbank" actors in the financial system, who are not members of either a settlements systems with netting like CHIPS or a settlements system with access to daylight credit from a central bank. They thus rely on their own liquidity to settle their obligations. A key issue I study, then, is how extensive this liquidity has to be to avoid gridlock.

There is also an interesting connection between this paper and the more abstract treatment of monetary exchange in Ostroy and Starr (1974). They consider a situation where a set of agents has a vector of endowments and must make a vector of net trades to achieve a Walrasian allocation. Ostroy and Starr (1974) show that having each agent barter once with every other agent is not generally enough to achieve this allocation, as long as the exchanges that agents have in each bilateral encounter do not rely on information about the history of other agents' trades. They prove that, by contrast, a single round of bilateral meetings is sufficient if each agent has an endowment of a "monetary" good whose value is large enough to cover the cost of all the purchases this agent requires. The setting considered here is related because each firm wishes to "repurchase" the coupons comprising its current obligations.[6] Nonetheless, I show that it is not necessary to give each agent enough money to pay all his obligations to ensure that full settlement takes place.

[5] In the case of Fedwire, Mengle (1985) notes that these daylight overdrafts came into existence because Fedwire regulations only required paying banks to have sufficient funds "at the end of the day." In the case of the Swiss system, the performance of the system without central bank liquidity provision was evidently unsatisfactory.

[6] I am grateful to Ivan Werning for pointing out this interpretation of my model.

The paper proceeds as follows. Section 1 contrasts the standard model of intermediaries where these channel funds from ultimate borrowers to ultimate lenders (see Diamond (1984) for a classic example) with a setting where there are debt “cycles” among intermediaries. A simple cycle would be a situation where a bank B lends to hedge fund A, which acquires claims on a financial firm C, which in turn uses its funds to lend to B. Such cycles emerge easily when financial firms, such as hedge funds, borrow from one firm while holding derivatives that impose financial obligations on another. To complete the cycle, the bank financing the hedge fund must also have a contract that obligates it to make payments to the hedge fund's counterparty. Section 2 then presents a more complex model of interconnected lending and considers the case where there is no limit on the number of times a unit of money can be reused within a period. Then, a central planner guiding payments can clear all debts with an arbitrarily small dose of liquidity. In the case of decentralized decisions, however, more interconnections require more liquidity if one wants to be sure that all debts clear.

Section 3 endogenizes the debt structure of Section 2. The purpose of this is to demonstrate a pecuniary externality that arises at the stage at which firms decide to whom they wish to extend loans. When a financial firm decides not to lend to another, this can easily reduce the interconnectedness of the financial system (since the second firm may well be forced to curtail its lending as well). This means that a firm's decision not to lend can increase the ease with which other firms settle their obligations in times where liquidity is short. Thus, the equilibrium degree of interconnectedness can be excessive from a social point of view.

Section 4 considers a setting where there is an exogenous limit on the number of times that a unit of liquidity can be used to settle obligations within a period. This may constitute a step towards realism relative to the case of potentially infinite chains of payments considered in section 2. This limitation on payments implies, in particular, that a larger volume of liquidity is needed to settle a larger volume of debt, even if interconnectedness is held constant. A more surprising result is that interconnections can now raise difficulties for settlement even if an omniscient central planner determines who pays whom. Section 5 offers some concluding remarks.

1 Setting the stage: Vertical lending versus debt cycles

A simple, and standard, view of financial intermediaries is that these channel funds from ultimate lenders to ultimate borrowers. As ultimate borrowers repay their obligations, intermediaries are able to repay their obligations to ultimate lenders as well. If contracts are simple and intermediaries have claims on borrowers that equal their liabilities to lenders, the capacity of all ultimate borrowers to repay their debts assures that all intermediaries are able to settle their own obligations as well. To see this, start with a trivial example where a lender has a claim of z against an intermediary, who in turn has a claim of z against a borrower. When the borrower repays the z that he owes, the intermediary is able to fulfill his obligation as well.

This result readily extends to more general situations where claims are "vertical," so that intermediaries only channel repayments from ultimate borrowers to ultimate lenders. To see this, suppose that there are N firms indexed by i. Let d^{ij} denote the amount that firms i is expected to pay firm j and let all these obligations be multiples of z.

It is helpful to notice that one can represent these obligations with a *directed graph* $\mathcal{G}$ where the vertices represent firms and where there is an "edge" going from vertex i to vertex j whenever i owes z to j. Debts that are multiples of z are represented by multiple identical edges. A certain amount of graph-theoretic nomenclature proves useful. The *in-degree* of a vertex is the number of edges that end at this vertex while its *out-degree* is the number of edges that originate from it. Further, a directed graph is connected if one can travel from any vertex to another by going along a series of edges, where travel always goes from the origin to the destination of the edge. When traveling in this way, a *cycle* denotes a set of edges that constitute a path from one vertex back to the same vertex.

A generalized "vertical" system of obligations can be represented by a graph that contains no cycles, and Figure 1 shows an example of such a graph. An acyclic graph must have some sources and some sinks, where the former are vertices with an in-degree of zero and the latter have an out-degree of zero. In the current context, sources (of payments) are net borrowers and sinks are net lenders. Intermediaries are represented by vertices that are neither sources not sinks, and Figure 1 displays three of them. It is apparent that intermediaries can have obligations to each other in such a graph. To avoid cycles, however, an intermediary that receives funds from a second has to effectively be funneling funds from that intermediary to a final lender.

I suppose that funds paid by one firm to another can be used by the receiver to make further payments "within the period." The idea is that obligations are due on a particular calendar day while money can be reused multiple times within the day. As the following proposition indicates, the result is that all obligations are met when firms are solvent and there are no cycles.

Proposition 1. *Let the graph describing the obligations of firms be acyclic. Then all debts are settled if firms are solvent.*

Proof. Solvency ensures that all sources are able to meet their obligations while also implying that any vertex that is neither a source or a sink has an in-degree equal to its out-degree. Let an edge from i to j be removed whenever i makes a payment of z to j so that full repayment of all obligations takes place when all edges are removed.

Start with the payment of z from a source to one of its creditors and remove the edges from this source to this creditor. Now let this particular creditor make a payment to one of his own creditors and remove the corresponding edge and continue making payments and removing edges until this particular payment reaches an ultimate lender. Next remove the edges associated with another payment from a source and, when this payment reaches an ultimate lender, continue in the same fashion until all sources have made all their required payments. It should be apparent that the graph contains no further edges. The reason is that any remaining edge would either have to originate from a source (which is impossible) or from a firm that still has an extant obligation that is traceable to a source (which is equally impossible). $\square$

Once financial firms move beyond vertical lending and trade claims against each other, cycles are likely to arise. A simple cycle is generated if, for example, firm 1 owes z to firm 2, who owes z to firm 3, who owes z to firm 1. These obligations leave all three firms "solvent" but unable to settle their debts without any outside source of liquidity. In this particular case, the needed liquidity might be obtained by inserting one of them into a vertical lending relationship. This is depicted in panel (a) of Figure 2, which combines the cycle I just described with a debt of z from B to 1 and a corresponding debt of z from 1 to L. Now, when B repays his debt, firm 1 can first repay firm 2, which repays firm 3, which then makes z available to firm 1 so that it can repay L. Thus, all debts can be settled by the simple device of giving firm 1 access to liquidity from outside the system consisting of $\{1, 2, 3\}$.

While this device can be effective, it is not infallible. Its success requires that firm 1 repay firm 2 before it repays L. In this simple case, it may seem obvious that this is in firm 1's interest. However, consider the simple variant depicted in panel (b) of Figure 2. Here, firm 1 does not owe funds to an identifiable ultimate lender L but to firm 4 who in turn owes funds to L. If firm 1 does not know the creditors of firm 2 and 4, he may sometimes pay firm 4 before he pays firm 2. It might then be necessary to give firm 1 additional sources of liquidity to guarantee that all debts are settled. This example demonstrates that, when there exist horizontal debt ties, full repayment by ultimate borrowers is no longer sufficient to ensure the settlement of all debts.

One potential way to proceed at this point would be to consider more general debt patterns that include both vertical relationships and cycles. Because the analysis becomes intractable quickly, I follow a simpler route and study how much "exogenous" money (or liquidity) is needed to settle obligations consisting exclusively of cycles. One source of exogenous liquidity is the holding of either monetary assets or assets that remain readily convertible into money even in a liquidity crisis. A second source consist of payments by ultimate lenders to financial intermediaries. These serve to pay off other financial firms as long as these funds are not funneled back in the direction of ultimate borrowers. The analysis that follows can thus be understood as one where some firms do have claims on outsiders and where this source of liquidity is used as much as possible because firms do not repay their obligations to agents outside the financial system until they have repaid all their debts to financial firms. One conclusion from this analysis is that, as financial firms become more interconnected, ensuring that one firm has access to outside liquidity may no longer be sufficient for settling all debts.

2 A settlement model with long payments chains

Consider an economy populated by N financial institutions (or firms) indexed by $i \in [0, 1, \ldots, N-1]$ and let these firms be arrayed in a circle so that firm $N-1$ is followed by firm 0. These firms have obligations that they are due within the period. In particular, each firm is expected to pay z dollars to the firms whose index is $i+j$ with $j \leq K$ where the addition is taken modulo N. Notice that, since each firm owes zK and is owed zK, this combination of debt and assets leaves each firm solvent. I denote the graph of these obligations by C_N^K. It has the property that the out-degree and in-degree of each of its N vertices is equal to K.

Using the interpretation based on Ostroy and Starr (1974) outlined in the introduction, one can think of each firm as wanting to achieve an allocation where it re-acquires the K coupons of its own debt while it sells back to the issuers the K coupons that it holds. As in their model, this vector of net trades is consistent with budget balance for each firm. As they argue, these net trades are easy to consummate with a single round of bilateral encounters between each pair of possible firms if each firm starts out endowed with a value of money greater than or equal to zK. Each firm then buys a coupon it has issued whenever it encounters a firm that holds one. The end result is that all coupons are purchased by their issuers.

In the Ostroy and Starr (1974) analysis, the sequence of encounters is not necessarily related to the transactions agents wish to carry out. Here I consider a somewhat more directed sequence of pairwise exchanges that is based on the idea that firms incur a penalty if they do not pay their debts on time. Thus, firms that simultaneously have money and debts would like to use their liquidity to make payments to at least one creditor. In other words, the structure of the problem suggests that firms know which firms they owe funds to, and that they wish to send funds to these firms if they can. This still leaves a firm with multiple creditors with the choice of whom to pay first.

How firms deal with this choice is an empirical question that deserves attention. Since firms are reluctant to reveal the identity of their creditors, this choice is unlikely to be based on knowledge of the entire system of outstanding obligations. My focus is on the extreme opposite case, where a firm knows the list of firms to whom it owes funds but knows (or remembers) nothing else about them.[7] As a result, they are indifferent regarding whom they pay. Moreover, I do not allow creditors to communicate with debtors before the latter choose whom to pay, so creditors are unable to affect this choice.[8] Lastly, I suppose that firms use whatever cash they have to pay off one of their obligations in full before seeking to pay any fraction of another obligation.

In practice, the process of consummating, verifying and recording payments does take some time, so there may be a finite upper bound R to the number of payments that can be made within the period using a single unit of liquidity. Below, I consider the case where the upper bound R is binding. I start with an even simpler case where each payment is processed so rapidly that R is effectively infinite.

To model the consequences of the finiteness of the supply of liquidity, I endow financial firms with units of liquidity in sequence. As soon as one firm receives some liquidity, it uses it to make payments, and the recipients of these payments make payments in turn. No further liquidity is injected into the system until the existing units of liquidity can no longer be used to settle existing obligations. At that point, another firm may receive a new endowment of liquidity. After these liquidity introductions cease, and when the existing liquidity can no

[7]An interesting open question is the extent to which the liquidity needed to settle all debts is reduced if firms have more information regarding the debts of their creditors. This information might lead firms to give priority to creditors with more obligations and thereby lead to a more efficient settlement mechanism.

[8]This assumption may appear restrictive. It is important to note, however, that creditors do not want to have a reputation for accepting less than the amount that is due to them. They may thus seek ways to commit themselves not offer debtors inducements to obtain payment ahead of other creditors.

longer be used to satisfy obligations, the settlement period ends. A firm i that still has open obligations at this point must pay a default cost c. This section focuses on the number of (sequential) distributions of liquidity from outside the financial system that are needed to settle all the debts and thereby avoid these costs.

Because firms do not favor one creditor over another, there are many possible paths of equilibrium payments. This leads me to study a best case scenario as well as a scenario that uses the available liquidity in the least effective possible way. The best case scenario turns out to be remarkably good: the minimum amount of liquidity that is needed to settle all debts is arbitrarily small and it is enough that one firm be endowed with this minuscule amount of liquidity.

To demonstrate this, it is worth recalling that an *Eulerian cycle* is a cycle that traverses every edge of a graph once, and does so in the direction of the edge. A graph is Eulerian if it has an Eulerian path. One elementary result in graph theory is that a graph is Eulerian if it is connected and each vertex has an in-degree that equals its out-degree. A second result that is relevant for this paper is that an Eulerian graph can be decomposed into cycles which are *edge-disjoint* so that these cycles do not have any edges in common. In the case of C_N^K, both the in-degree and the out-degree of each vertex equal K. Moreover, the graph is connected since one can always reach vertex j from vertex i by traveling to $i+1$, $i+2$ and so on until one reaches j (by passing vertex 0 if $j < i$). This graph is thus Eulerian.

Proposition 2. *Let firm i be endowed with an arbitrarily small amount of liquidity w. Using just this liquidity, a path of payments can be found such that all debts in C_N^K are settled within the period.*

Proof. Let $\hat{C}^i = \{i, j, k, \ldots, i\}$ be an Eulerian cycle originating at i. Suppose first that $w < z$. Then let i give w to j to settle part of his debt with him, let j use these funds to pay part of his debt to k, and so on along the Eulerian path until these funds reach i. At this point, every firm owes $z - w$ to its K creditors. If this exceeds w, i once again pays w to j and so on along the Eulerian path. When enough Eulerian cycles of payments have been completed that everyone's outstanding debt $\tilde{z}$ is less than w, i pays $\tilde{z}$ to j who passes it on to k, and so on, until all debts are settled. This last case also applies when $w \geq z$. $\square$

While the particular graph considered in this proposition is special, it is clear from the proof that the result applies to any pattern of debts that can be represented by Eulerian graph. As long as all financial firms are connected to one another, the graph of their obligations has this property whenever each firm has total obligations to other financial firms that equal its total claims on such firms. Since the model neglects the connections of financial firms with ultimate borrowers and ultimate lenders, this is an automatic consequence of supposing that financial firms are solvent. Solvency would remain sufficient to guarantee this condition if, as in panel (a) of Figure 2, financial firms that owe funds to ultimate lenders have equal claims on ultimate borrowers. In practice, these claims are probably unequal for many firms, so solvency does not imply an Eulerian graph of debts among financial firms.[9] Nonetheless, the basic implication of this proposition, that a small amount of liquidity provided from outside the financial system (by ultimate borrowers, for example) is sufficient for the financial system to settle all its debts, may well carry over to this case.

This proposition shows that the best case scenario requires very little liquidity to settle all debts. The Eulerian path(s) that accomplish this are straightforward to compute for a central planner with full information regarding everyone's debts. In a decentralized system with the limits on information that I have imposed, however, much more liquidity may be needed. To show this, I now focus on a case that is the "worst case scenario" from the point of view of how liquidity is used to settle debts.

Imagine that, as in the proof of Proposition 1, the edge from i to j is removed when i pays his obligation to j. Consider the initial graph C_N^K and give an endowment of z to i. As these funds flow from firm to firm, they remove edges until i has no further obligations. At that point, i's vertex can be removed as well and i's liquidity endowment is back in his hands. The return of liquidity to the firm that originally obtained it follows from the fact

[9] If vertices corresponding to ultimate lenders and ultimate borrowers are included in the graph, the graph is obviously not Eulerian, since the in-degree and out-degree are not equal for these vertices.

that each firm has the same number of debts as it has claims on other firms. This implies that, whenever a firm j (not equal to i) receives z as payment, j still has a debt that it can extinguish by paying z to yet another firm. As a result, any unit of liquidity with which i is endowed continues to be used for payments until it is back in the hands of i himself. As long as firm i still has obligations, it makes further payments and this implies that payments continue until i has settled all his obligations and is in possession of his initial endowment. If there were an active market for intra-day credit, firm i could lend these funds to another firm that still needs to make payments. However, I am focusing on situations were such loans are impossible.

Firm i is part of K distinct cycles in graph C_N^K, so i's endowment must travel through K cycles before it stops being useful for payments. To keep the total number of payments made with this endowment to a minimum, these cycles must be as short as possible. This idea that payments follow the shortest possible cycles is captured in Assumption A, which also covers subsequent endowments. In particular, let G_t denote the graph that is left after t firms have each been given an endowment of z and made all the payments that this endowment facilitates, with $G_0 = C_N^K$. Then,

Assumption A. If firm i receives an endowment when the graph of obligations is G_t, the path followed by his first payment follows one of the shortest cycles in G_t that includes i. If at any vertex j of this cycle (including the origin i) there is more than one shortest path back to i, the one that is chosen is the one that maximizes z where the edge $\{j, j+z\}$ is included in the cycle. If i still has outstanding debts after earlier payments return to him, he makes new payments. These are chosen in a like manner.

The inefficient use of liquidity in Assumption A is counteracted to some extent by Assumption B, which concerns how the sequential endowments are distributed. Let d_t^j represent the total obligations of firm j at t, with $d_0^j = zK$. Suppose that, at stage t, there still exists a firm i such that $d_t^i > 0$. Then,

Assumption B. If a firm i receives an endowment after t firms have received theirs (and made all possible payments), $d_t^i \geq d_t^k$ for all k between 0 and $N-1$.

The purpose of Assumption B is to ensure that liquid endowments go to the firms that need them the most (because they have the largest debts). The reason to make this assumption is that, without it, it is easy to waste massive amounts of liquidity by giving it to firms that have already settled all their obligations in the past. It does not seem reasonable to compute the minimum amount of liquidity needed by the system while allowing large amounts of liquidity to be wasted in this manner.

Proposition 3. *Under Assumptions A and B, the minimum number of firms that must be provided with liquidity to settle all obligations in $G_0 = C_N^K$ is K.*

Proof. Start by giving z to firm i. The K shortest cycles starting at i on C_N^K start at $i+j$, $1 \leq j \leq K$, then go to $i+j+K$, $i+j+2K$ and so on, until they reach $\{i-K, \ldots, i-1\}$, at which point they return to i (where all these numbers are modulo N). These K cycles are edge-disjoint and the full set of them touches each vertex once. Once the edges that are part of these cycles are removed, one can remove vertex i as well since i is left with no debts or claims. This leaves the graph G_1 which is given by C_{N-1}^{K-1}. In this graph, each vertex has $d_1^j = K-1$.

Assumption B implies that one of these remaining firms receives the next unit of endowment. By the argument above, G_t is thus C_{N-t}^{K-t} for all $t \leq K-1$. After $K-1$ firms have been given an endowment, the graph is C_{N-K+1}^1. Denoting the K'th firm that receives an endowment by i, this firm pays $i+1$, who pays $i+2$ and so on until all the debts are cleared. □

This proposition shows both that giving K separate firms an endowment is enough to clear all debts and that, under assumptions A and B, giving endowments to fewer firms leaves some firms unable to settle their obligations. Indeed, if only $K-1$ firms are given an endowment, only $K-1$ firms clear their debts and the remaining $N+1-K$ firms are unable to do so. This shows that an increase in the interconnectedness of firms increases the liquidity that is needed to settle all debts under assumptions A and B.

Even though these assumptions imply that more liquidity is needed than in the best case scenario where payments follow Eulerian paths, the required liquidity zK is still smaller than the sum of all obligations, which equals zKN. This is worth noting because Ostroy and Starr (1974) only show that it is sufficient for each agent to have as much money as his total purchases, which here corresponds to giving each of N firms an endowment of zK. The reduced liquidity required in my setting is not the result of assuming that each firm has more bilateral exchanges with other firms. Ostroy and Starr (1974) suppose that, in one "round," each agent encounters every other agent once. Both in the Eulerian path and in the paths contemplated in Proposition 3, each firm pays each of its creditors only once so that the number of pairwise meetings is actually smaller. It thus appears that Ostroy and Starr (1974) could have found a tighter bound on the amount of money needed to achieve their desired outcome. Still, the random sequence of meetings envisaged by Ostroy and Starr (1974) may require more liquidity than the sequence I consider here, where meetings are initiated when a debtor is able to make a payment to a creditor.

As the following proposition demonstrates, zK units of liquidity can settle an even larger volume of total debt when there is no restriction on the number of bilateral exchanges.

Proposition 4. *If debts have the pattern embodied in C_N^K, giving endowments of z to K firms under assumptions A and B clears all debts even if each firm's bilateral obligation z_o exceeds z.*

Proof. Let q equal z_o/z when this ratio is rounded down and let $z_r = z_o - qz$. Start by instituting a cycle of payments that would clear all debts if $z_o = z$. Then repeat this path of payments an additional q times, where the first q such cycles transfer z each and the last transfers z_r. □

This proposition also implies that z, and thus the total size of debts, does not affect the minimum amount of liquidity needed. Only the interconnectedness of debts K matters when R is sufficiently large.

To gain intuition for the model and its behavior, consider the simple case shown on Figure 3 where $N = 6$ and $K = 1$. Suppose that only firm 0 starts out with liquidity equal to z, perhaps because it is the only one involved in a vertical lending chain like the one depicted in panel (a) of Figure 2. Within the financial system, this firm has no one to pass this liquidity to other than firm 1, who passes it on to firm 2 and so on until firm 5 returns it to 0. In the process, all debts are settled. Figure 3 also makes it clear that this result does not depend on N being equal to 6: no firm has a choice as to whom to pay when $K = 1$, so all payments complete a full circle before returning to firm 0.

This can be contrasted with Figure 4 where the left panel shows C_6^2. The middle panel shows an Eulerian cycle. In this cycle, 0 makes a second payment after the funds it has advanced first get returned to him. Suppose that the first payment is given by the dashed arrows so that the vertices it reaches, in order, are $\{0, 2, 4, 5, 1, 3, 5, 0\}$. The second payment then follows the solid arrows so that its path is $\{0, 1, 2, 3, 4, 0\}$. When all these payments have been made, all obligations have been settled. Note that it is crucial for this particular Eulerian path to be completed that 5 first pass to 1 and only later pass to 0. The right panel shows a less happy outcome where the first of 0's payments follows $\{0, 2, 4, 0\}$ so that 4 passes immediately to 0, while the second payment follows $\{0, 1, 3, 5, 0\}$, so that 5 passes to 0 at his first available opportunity. The multiplicity of choices faced by each firm in the case $K = 2$ makes it easier to construct paths of payments such that giving liquidity to just one firm is insufficient to settle all debts. As K is increased further, this multiplicity can be exploited so that even more firms must be given liquidity.

So far, this section has only considered the fully symmetric graph of obligations C_N^K. To study whether firms make optimal decisions when they acquire claims and debts, however, one must study what happens when one firm has fewer assets and liabilities. It is, of course, impossible to reduce *only one* firm's obligations since eliminating i's obligation to j means that j is unable to pay off as many obligations as before. If j responds by reducing his obligations to k, firm k must further reduce his own debts. This logic implies that, starting with the graph C_N^K, at least one cycle must be removed for i to have one fewer obligation while ensuring that all firms still have the same number of claims as they do debts. Consider then, the graph $\mathcal{G}_i = C_N^K - C_i$ where C_i is a cycle that passes through i.

Intuition would suggest that, since there are fewer debts to settle with $\mathcal{G}_i$ than with C_N^K, complete settlement of all debts can be accomplished by providing fewer firms with liquidity in the former case. This can be seen graphically for a special with $N = 6$ and $K = 2$ in Figure 5. In this Figure, one cycle has been removed from C_6^2, namely the cycle given by $\{1, 3, 5, 1\}$. Inspection of the Figure shows that giving a liquid endowment to any of the firms with two debts (0, 2 or 4) is enough to clear all debts because these firms first make a payment that travels along the dashed arrows and then make a second payment that travels along the solid ones.

Using numerical methods, it is readily shown that the basic conclusion from this example extends to other values of K and N. I have considered a range of values for these parameters and constructed $\mathcal{G}_i$ by subtracting a shortest cycle from C_N^K. In other words, I subtracted a cycle such that all but one of its edges went from a vertex with index i to a vertex with index $i + K$, while the remaining edge went from a vertex with index i to a vertex with index $i + r$ where r is the remainder in the division of N by K. I then assigned endowments using Assumption B and chose paths of payments consistent with Assumption A. When a firm receives an endowment, Assumption A uniquely determines these paths. By contrast, Assumption B does not uniquely determine which firm receives an endowment from among all the firms that have the maximum total debt. In the case of C_N^K, this ambiguity was not important because all firms were symmetrically placed after an endowment had been used as much as possible for payments. In the case where one cycle is removed from C_N^K, however, firms are not as symmetric. The numerical analysis reveals that, as a result, the total number of firms that must be given liquidity to settle all the debts depend on the identity of the particular firms that are given liquidity. While I did not study this dependence exhaustively, many possible allocations were considered and, in all cases, fewer than K firms had to be given liquidity to settle all debts.

3 A model of multilateral claims acquisition

To show the effect of interconnectedness on the amount of liquidity that might be needed for settlement, one can treat the level of obligations as exogenous, and this is the course pursued in the Section 2. This analysis leaves open, however, whether the interconnections that are observed in equilibrium are excessive or not. To show that the equilibrium degree of interconnectedness need not be socially optimal, this section develops a very simple model of claims acquisition. Unlike the settlements process considered in the previous sections, the acquisition process is carried out without the use of outside liquidity. The implicit idea is that, while claims are being acquired firms can simply exchange IOU's with one another and use other firms' IOU's to acquire claims that they find more attractive. This trust in IOU's disappears in the periods of liquidity shortage that give rise to my study of settlements.

The particular model I consider is somewhat unusual both in the way that it creates demands for securities and in the centralized mechanism that it postulates for determining who holds claims on whom. It tries to capture two fairly conventional forces, however. The first is that firms differ in the claims that they wish to hold. The second is that financial intermediaries have an incentive to maximize the volume of intermediation.

One reason why people and firms may wish to hold different portfolios from one another is that they differ in the returns that they expect from different securities. Models where people differ in their equilibrium beliefs are somewhat complex, however, and I thus opt for a simpler approach that relies on "tastes." In particular, firm i is assumed to derive utility $u(j)$ from holding a claim of z on firm $i - j$. Claims smaller than z yield no utility, and neither does utility rise if the size of claims is increased above z. This extreme concavity leads firms to be unwilling to lend more than z to anyone and this fits with the common tendency of many financial market participants to limit their exposures to individual counterparties as a method to manage their counterparty risk (see Corrigan, Theike et al. 1999, p. B1 for a description and discussion). I let $u(j)$ be decreasing in the index j so that firms have an intrinsic preference for holding the claims of firms that are close to them when going in the direction where the firm index falls. There is an extensive literature demonstrating that people and firms' portfolios contain relatively large proportions of claims on "local" creditors, and the model is partially faithful to this effect by giving firms a preference for claims whose indexes are close to their own.

Explicit modeling of a decentralized system where individuals have something to gain by arranging trades by third parties is also beyond the scope of this paper, even though the issue occupies a central role in the financial services industry. I postulate instead a centralized mechanism whose aim is to maximize financial transactions on the basis of messages sent by participating firms. The message sent by firm i consists of the integer ℓ_i. This integer is interpreted as the number of firms that i is willing to lend to if it has the resources to do so. Because i is known to have a preference to lend to local firms, the message is taken to mean that i is willing to lend resources to all firms whose index is $i-j$ where $1 \leq j \leq \ell_i$ and the subtraction $i-j$ is modulo N.

On the basis of these messages, the mechanism determines the matrix X whose element X_{ji} is equal to 1 if firm i lends z to j and equals zero otherwise. The ith column thus indicates the firms to whom i lends funds, while the jth row indicates all the firms that lend resources to j. Letting ι represent a vector of N ones, the requirement that each firm's total loans be equal to its total obligations can be written as

$$X\iota = X'\iota \tag{1}$$

where X' is the transpose of X. Thus, the sum of the elements of a row is equal to the sum of the elements of the corresponding column. With X_{ij} only being able to take the values of zero and one, the centralized mechanism maximizes the total value of claims

$$\iota' X \iota \tag{2}$$

subject to (1) and

$$\forall i, j \qquad X_{ji} I_{ji} = 0 \qquad \text{where} \begin{cases} I_{ji} = 0 & \text{if} \quad \ell_i \geq i - j \\ I_{ji} = 1 & \text{otherwise} \end{cases} \tag{3}$$

This constraint ensures that firm i does not hold a claim on a firm that is further than ℓ_i away from it.

Letting ℓ denote the full set of messages, the solution to this optimization problem is the matrix $X^*(\ell)$. The matrix X^* is the *adjoining matrix* of a directed graph, since it has zeros on the diagonal while some of its off-diagonal elements equal one. Since X^*_{ij} is equal to one when i owes funds to j, and since this debt contract requires i to pass z units of liquidity to j, X^* is in fact the adjoining matrix for a settlements graph.

From the perspective of firms i, it is useful to decompose ℓ into the message sent by i himself, ℓ_i and the messages sent by all other firms ℓ^i. Firm i then chooses ℓ_i to maximize his own utility, which is given by

$$U_i = \sum_{j=i+1}^{i+N-1} u(i-j)X_{ji} - P_i(X^*)c$$

where $P_i(X^*)$ is the probability that firm i will be unable to settle one of its obligations given the debts represented by the matrix X^*. The maximization of this utility requires individuals to have beliefs about the effect of ℓ_i on i's assets and liabilities as well as on i's probability of being unable to meet his obligations. I require that these beliefs satisfy rational expectations. This means that firm i has to know both the equilibrium value of $P_i(X^*)$ as well as how this probability changes when ℓ_i changes. Notice that the rational expectations assumption does not require firms to know the full network of obligation, nor their position in this network. Asking for such knowledge would seem unreasonable in more realistic settings.

I focus on symmetric equilibria where all firms send a message $\ell_i = \bar{\ell}$. Given such symmetric messages, the maximization of $\iota' X \iota$ leads the pattern of obligation to be equal to $C_N^{\bar{\ell}}$ and thereby reproduces the debts considered in the previous section. Assuming that $\bar{\ell}$ firms chosen sequentially according to Assumption B are given endowments of liquidity and that settlements proceed according to Assumption A, $P_i = 0$. Under assumptions A and B, these default probabilities are higher if the number of firms that receive liquidity has a positive probability of being smaller than $\bar{\ell}$.

For a symmetric equilibrium to exist, no firm must want to unilaterally deviate from sending a message of $\bar{\ell}$. When a single firm deviates by setting ℓ_i above $\bar{\ell}$, X^* is unaffected. Since the mechanism limits the loans of all other firms to $\bar{\ell}$, firm i does not have the resources to increase the number of his loans beyond this. The ineffectiveness of a message that is above that of all other firms implies that firms cannot gain or lose from sending messages that are above the consensus message $\bar{\ell}$. This indifference could justify assuming that firms send messages of $\bar{\ell}$ whenever they believe that other firms do so, even if all firms preferred to make loans to more firms. This could then rationalize equilibria with arbitrarily small (and even zero) loans. Such equilibria are not robust, however, since they hinge on reacting to indifference in a very particular way. They are also unattractive because they ignore the efforts of financial intermediaries to coordinate their actions when this is profitable.

I thus center my attention on symmetric equilibria where firms are indifferent with respect to reductions in ℓ_i. A reduction in ℓ_i below $\bar{\ell}$ does affect equilibrium lending because it prevents the centralized mechanism from giving firm i claims on $\bar{\ell}$ firms. Indeed, (1) requires a reduction also in the number of firms that lend to i and in the loans of at least some of the firms to whom i would have lent if ℓ_i had been set equal to $\bar{\ell}$.

Consider then, a deviation where $\ell_i = \bar{\ell} - 1$. Because i can end up with at most ℓ_i claims and obligations, the resulting X^* must feature at least one less cycle passing through i than the graph $C_N^{\bar{\ell}}$. Since the mechanism seeks to maximize the number of edges remaining in X^*, it removes a shortest cycle. As discussed in the previous section, this implies that endowing $\bar{\ell} - 1$ firms with liquidity is then sufficient to settle all debts under assumptions A and B.

The aim of the current section is only to demonstrate that the acquisition of claims need not be optimal. I thus proceed to construct a special case where private and social interests diverge, with the hope that it provides some intuition that is more generally valid. Suppose that assumptions A and B hold, that it is certain that at least $\bar{K} - 1$ firms will receive endowments of liquidity and that there is a probability μ that $\bar{K}$ firms will do so. I now consider a sufficient condition for an equilibrium to exist such that all firms set ℓ_i equal to $\bar{K}$.

At such an equilibrium, all debts are settled with probability $(1-\mu)$. With the remaining probability, $N-\bar{K}+1$ firms are left with one unpaid debt. The remaining $\bar{K}-1$ firms settle all their debts because they are the lucky recipients of a liquidity endowment. So, the combination of not knowing how many units of liquidity will be available and not knowing which firms will receive liquidity in the case where only $\bar{K}-1$ units are available leads firms to have an expected default cost of $\mu c(N-\bar{K}+1)/N$.

A firm that deviates from the proposed equilibrium by setting $\ell_i = (\bar{\ell}-1)$ avoids these default costs since it is certain to be able to fulfill all its obligations. Since this deviation costs the firm $u(\bar{K})$, it is indifferent with respect to this deviation if

$$u(\bar{K}) = \frac{\mu c(N-\bar{K}+1)}{N} \tag{4}$$

Condition (4) ensures that there is an equilibrium with $\bar{\ell} = \bar{K}$. Symmetric equilibria with smaller numbers of loans also exist if all firms set ℓ_i to smaller values. What is less appealing about these equilibria is that all firms prefer to have more debts, so their existence relies on firms sending the message $\bar{\ell}$ rather than $\bar{\ell}+1$ only because they are sure that it will make no difference. To see this, consider an equilibrium with $\bar{\ell} = \bar{K}-1$. If firm j thought that it stood a chance of obtaining $\bar{K}$ debts and assets by sending a message of $\bar{K}$, it would do so. His benefit from doing so would be $u(\bar{K})$. His loss, meanwhile, would be $\mu c(N-\bar{K}+1)/N$ if every other firm sent a message of $\bar{K}$. If fewer firms did so, but nonetheless enough of them did it for firm j to end up with $\bar{K}$ debts and assets, Assumption B ensures that firm j would have a greater than $(\bar{K}-1)/N$ probability of being a recipient of a liquidity endowment. The deviating firm would then be assured of settling its debts even if only $\bar{K}-1$ receive an endowment. This firm thus stands some chance to gain, and no chance to lose by sending a message of $\bar{K}$.

For allocations with even lower values of $\bar{\ell}$, a single firm is strictly better off if sending a message of $\bar{\ell}+1$ leads it to acquire $\bar{\ell}+1$ debts and assets. At these lower levels of indebtedness, all debts are settled with probability one, so that the firm simply gains $u(\bar{\ell}+1)$ if it succeeds in increasing the size of its balance sheet.

I now study the social consequences of having firm i reduce ℓ_i from $\bar{K}$ to $\bar{K}-1$. For a certain number of firms, this reduces the number of their debtors and creditors by one. Given that the mechanism maximizes total debts, the number of firms thus affected is N/K rounded up to the nearest integer. These firms all lose $u(\bar{K})-\mu c(N-\bar{K}+1)/N$ so that they neither gain or lose anything. For the rest of the firms, there is a net gain of $\mu c(N-\bar{K}+1)/N$ since their debts are now settled for sure. To obtain the total social gain, one multiplies this individual gain by the number of these indirectly affected firms, which is $(N-N/\bar{K})$ rounded down. The reason these social gains exist is that Assumption A implies that liquidity is not used in its most socially efficient manner. As a result, reducing a few firms' liquidity requirements allows many other firms to take advantage of the liquidity that is thus freed up.

One possible interpretation of the social benefits of having a firm reduce its interconnections is that each firm's financial transactions create a "congestion externality" in that it makes a demand on scarce (and unpriced) liquidity. This congestion externality is unusual, however, in that small reductions in a firm's interconnections can have discrete benefits for a great many other firms. The reason is that even a small reduction in interconnections can eliminate some liquidity payments paths that are quite inefficient, and thereby allow the existing liquidity to settle many *more* debts. This can be seen in Figure 5 where the elimination of the cycle of obligations $\{1,3,5,1\}$ implies that, under Assumptions A and B, all debts are settled with just one unit of liquidity. By contrast, when the $\{1,3,5,1\}$ cycle remains present, one unit of liquidity is not enough.

4 Short payments chains

There are several reasons to be interested in situations where there are limits to the number of payments that can be settled by a unit of liquidity. One might suppose, for example, that the processing of each payment takes a discrete amount of time τ while the length of the trading day is itself limited and equal to T. It is then impossible to use a unit of liquidity for more than T/τ payments on a given calendar day and this may affect the amount of liquidity that one needs to settle the debts that come due on that day. As one firm is paying a second during a particular time interval, a third firm might be able to learn that it will receive the resulting funds later on. This third firm may thus be both able and willing to make a nearly simultaneous payment to a fourth firm using funds raised though a "daylight" loan. This parallel processing of payments may allow a unit of liquidity to be used more than T/τ in a given day.[10]

Nonetheless, there may well be limitations on the process of making payments in advance of receiving liquidity. One of these is that, when a bank's daylight loan is repaid, the bank receives liquidity. This liquidity can only be used to settle more debts if the bank lends it anew. If the bank fails to do so, only the original cascade of payments using the system's actual liquidity continues unabated.

This section thus takes up the case where the maximum number of times that a unit of liquidity can be used, R, is smaller than $N-1+K$ so that the paths of payments considered in Section 2 are infeasible.[11] One immediate consequence of this is that the total liquidity that is needed to settle all debts now depends on the volume of debt in addition to depending on the number of interconnections among firms. To see this, imagine a pattern of liquidity endowments that settles all debts when each firm owes z to each of his creditors. If each bilateral debt is of size λz (so that the total debt is multiplied by λ), it can be settled by the same sequence of endowments, as long as each endowment is multiplied by λ as well. Conversely, if one multiplies every bilateral debt by a sufficiently large λ, the original distribution of liquidity endowments will be insufficient to settle all debts.

A somewhat more surprising consequence of short payment chains is that the *minimum* number of liquidity endowments that is needed to settle all obligations can now depend on the interrelatedness of obligations. This can happen even if a social planner gets to

[10] If these loans are costly, firms would prefer to pay with cash that they have already received, and this might dampen the use of this borrowing. See Angelini (1998) for a model where priced intraday credit leads firms to postpone their payments until they have cash on hand.

[11] That the solution considered in Proposition 2 involves $N-1+K$ payments when the first receives z units of liquidity can be seen by noting that this unit of liquidity ensures that $N-1$ firms receive one payment while the original recipient of liquidity receives K payments.

determine the payment paths, so that interrelatedness poses problems even leaving aside the externality from choosing whom to pay (though this externality can still increase the number of firms that must be given liquidity in worst case scenarios). This new problem arises only when individual liquidity endowments are lumpy, so that some firms have large endowments of outside liquidity while others have no such endowments. What happens, then, is that interrelatedness makes it more difficult to channel large quantities of liquidity to firms without endowments.

To see this, I start with a case where endowment distributions are so small that N of them are needed even if $K = 1$. I then show that N distributions can also lead all obligations to be settled when K is larger, though this requires that payment paths be chosen with great care. This irrelevance of K when payment paths are chosen by an outside planner harks back to the irrelevance of K when long payment chains were forced to move along Eulerian paths. I then show that this irrelevance stops being true when endowment distributions are larger.

Suppose that each firm has a total debt d, where this debt is independent of K so that one can study the effect of debt interconnections. Then, if each endowment can be used R times and the size of each endowment is e, the minimum number of endowments that must be distributed equals dN/eR. If $e = d/R$, this minimum equals N and this value of e lets a planner achieve full settlement with minimum liquidity for many values of K.

Proposition 5. *If each endowment has size d/R and payment paths are chosen appropriately, N distributions are sufficient to cover all debts for any K if either R/K or K/R are integers.*

Proof. First let R/K be an integer r. The obligation z of firm i to firm $i+1$ is then equal to $d/K = rd/R$. Let each firm receive one distribution of d/R. To specify the jth payment made with a particular distribution let j be written as $j - 1 = mK + f$ where m equals $(j-1)/K$ rounded down and f is the remainder from this division. Then, the jth payment made with the endowment given to i goes from $O_{ij} = i + m\sum_{v=1}^{K} v + \sum_{v=1}^{f} v$ to $O_{ij} + f + 1$.

This means that, for $j < K+1$, the first payment travels a distance of 1, the second a distance of 2, and so on until the Kth payment travels a distance of K. The $K+1$st payment then again travels a distance of 1, followed by a payment that travels a distance of 2 and so on.

The edges corresponding to these payments cover the full edge set of the graph. To see this, note that, for $j \leq K$ firm i must make r payments of d/R to firm $i+j$. Moreover, this procedure ensures that it has enough resources to do so. To see this, notice that the closest source of these payments is the endowment given to the firm that is separated from i by $\tilde{O}_j = 1 + 2 + \ldots + (j-1)$. If $r > 1$, there are $r-1$ additional sources which are given by firms that are separated from i by $\tilde{O}_j + mK(K+1)/2$ where m takes values between 1 and $r-1$.

Now consider the case where K/R is an integer r' greater than 1. The obligation of firm i to firm $i+1$ can now be written as $d/K = d/(r'R)$ so that the firm is able to make r' of its required payments with his endowment of d/R. Let these payments be made to firms with indices $i+v$ where v goes between 1 and r'. Further, let firm $i+v$ make a payment to firm $i+2v+r'$, with subsequent payments going to firms with indices $i+kv+k(k-1)r'/2$ with k taking on values between 1 (the original recipient) and R. Notice that the last of these payments is made by the firm with index $i+(R-1)Rr'/2$ and made to a firm with index $i+(R+1)Rr'/2$, *i.e.*, a firm that is $Rr' = K$ firms away from the firm with the endowment. Thus, each endowment induces K payments, one of each possible length. Moreover, if the payment of length j made by firm i is induced by the endowment given to firm k, the endowment of firm $k+1$ leads to a payment of length j made by firm $i+1$. Thus, the N endowments given to the full set of firms lead all payments of length j to be made. $\square$

This demonstrates that the minimum of N distributions is achievable for both small and large values of K when the distributions are small enough that it is necessary to make N of them. In the case of $K = 1$, firms have no choice as to whom they pay. As a result, distributing d/R to each firm ensures that all firms make their required payments. In the case where $K > 1$, however, firms do face such a choice and the result of Proposition 5 hinges crucially on making these payment choices in a way that uses the endowments efficiently. To see this, I consider a simple example where the obligations are given by C_6^2, which is given in the left panel of Figure 4. Proposition 5 shows that 6 distributions of $d/2$ can be sufficient to settle all debts when $R = 2$. I now show that, for a different set of payment choices, 6 distributions of $d/2$ are not sufficient.

Start by giving endowments to firms with index i equal to 0, 2, and 4. let each of these firms make a payment of $d/2$ to firm $i + 1$, which then makes a payment to firm $i + 3$. The graph of remaining obligations is now given by Figure 6. It is apparent by inspection of this Figure that it is now impossible to make 3 further distributions of $d/2$ that settle all obligations. After giving endowments of $d/2$ to firms 1 and 2, for example, there remains an obligation from 0 to 2 and an obligation of 5 to 1, and these cannot both both be settled by distributing one endowment. Thus, the externality of payment choice remains present in this example with shorter payment chains.

A more surprising aspect of short payment chains is that they create problems for settlements in interconnected financial systems even when a benevolent planner chooses who makes payments to whom. These problems only arise, however, when endowments are lumpier than in Proposition 5. This is demonstrated in the following proposition.

Proposition 6. *Let each of N firms have a total debt equal to d and be owed d by others. Suppose that N is divisible by $R > 1$ and that liquidity endowments are equal to d. Then, the minimum number of liquidity endowments needed to clear all debts when these are given by C_N^1 equals N/R. When debts are given by C_N^K with $K \geq R$, the number needed is strictly larger.*

Proof. See Appendix. □

The proposition suggests that settling all obligations is more complex when $K \geq R$. When $K = 1$, it suffices to space the recipients of exogenous liquidity so that their indices differ by R. By contrast, when $K \geq R$ so that each firm makes payments to a variety of firms, each endowment leads many firms to have their debts reduced slightly. It then becomes impossible to space endowments so that each is received by a firm that has not been able to pay some debts with funds received from other firms. Thus, endowments are not fully used for payments and more firms need to be endowed with funds to settle all debts.

This result relates more to the technology of payments systems than to the behavior of the people operating in such systems. Nonetheless, it is worth trying to provide some intuition for it. In a system where everyone has the same total debt, full settlement requires that every firm receive the same amount of liquidity either from others or from agents outside the financial system. When each firm owes funds only to one other firm, the capacity of one firm to make all its required payments ensures that those firms that receive these funds are also able meet their full obligations. It is thus relatively easy to engineer a scheme where giving liquidity to just a few firms avoids all defaults.

When each firm has multiple creditors, by contrast, the capacity of one firm to meet all its obligations only ensures that many other firms can cover a small fraction of their obligations. This implies that even relatively few outside endowments lead almost every firm to be able to pay some of its creditors. This may seem like a benefit of interconnections, but the fact that each lumpy endowment only gives small amounts of liquidity to many other firms means that it is much harder to avoid "holes," *i.e.*, situations where many firms can cover only a fraction of their payments. To cure this, more firms must be given access to outside liquidity.

Proposition 6 covers the case where $K \geq R$ so that interconnectedness is large relative to the length of payment chains. This leaves open the question of what happens when $1 < K < R$. As the following examples indicate, it seems difficult to say much about this situation in general. When $R = 4$ and $K = 3$, it is sometimes possible to construct payment chains so that N/R endowments suffice (the example below shows this to be true for $N = 12$ but the result ought to apply whenever N is divisible by R). By contrast, when $R = 4$ and $K = 2$, there do exist values of N that are multiples of R such that N/R is not sufficient. This suggests that the capacity to settle all claims with N/R endowments is not monotonic in K when $K < R$.

Proposition 7. *When the graph of obligations is given by C_{12}^3 and $R = 4$, it is possible to clear all debts by giving 3 endowments of d.*

Proof. See Appendix. □

Proposition 8. *When the graph of obligations is given by C^2_{12} and $R = 4$, it is impossible to clear all debts with 3 endowments of d.*

Proof. See Appendix. □

5 Conclusions

The paper shows that limits on the amount of liquid assets that are available to the financial system for the payment of debts can lead to more defaults when the web of debts is more densely interconnected. Under the model's assumptions, a program of intra-period government lending for settlement purposes would eliminate defaults and thereby raise economic efficiency at no cost to the government. One key assumption that makes this possible, however, is that all firms have claims that are at least as large as their obligations so that a program of temporary lending to all firms does not make losses.

The purpose of this assumption, however, was not to argue that it is realistic to imagine a situation where firms are sure to be certain while, nonetheless, they are unable to get private loans that would be repaid with probability one. Rather, the purpose of the assumption is to show that fear of insolvency (which presumably stands behind the unwillingness of the private sector to make the necessary settlement loans) can create liquidity problems that are exacerbated by interconnections. These liquidity problems can lead to defaults even in those states of nature where firms do turn out to be solvent.

If one takes this perspective, a program of government lending would costlessly raise welfare in some states of nature (when firms are in fact solvent) but could incur costs in other states. To evaluate such a program, one would have to have an estimate of these costs, and this would in turn raise the question of whether the private sector's unwillingness to provide equivalent loans is based on a proper assessment of the costs of doing so. This paper is, however, completely silent on this issue.

To simplify the analysis, the model assumes a great deal of symmetry, and much of the analysis involves firms that have to pay the same amount to the to the same number of firms. This symmetry allows me to be somewhat silent concerning the maturity of the debts involved. One can interpret the required payment as the coupon on a long term debt (so that each firm's debt is expected to be unchanged when the period is over) or as principal plus interest on short term debt (so that firms are massively reducing their exposure to one another). In the latter case, required payments are obviously much larger for a given market value of total debt so that more liquidity is needed if payment chains are limited in length.

The model would be more realistic if it involved less symmetry, as well as if it simultaneously incorporated firms' vertical debt relations with borrowers and lenders outside the financial system itself. It is important to stress, however, that considerable care will have to be employed when generalizing the model in these directions to maintain analytic tractability. To get an idea of the distance that separates what has been demonstrated analytically for related graphs when the interconnectivity parameter K is varied and the sort of conjectures that experts regard as plausible, the reader is referred to Alon *et al* (1996).

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Appendix: proofs of some propositions

Proof of proposition 6

In the case of C_N^1, it suffices to give endowments to firm with indices given by iR with $i = 0, \ldots, N/R$ to clear all debts. Each recipient of an endowment i uses the funds to pay off his entire obligation to $i+1$. Payments from one firm to the next continue until firm $i+R$ receives a payment from the endowment given to firm i. Firm $i+R$ also receives an endowment and thus all obligations are cleared.

For the provision of d units of liquidity to N/R firms to be sufficient to settle all debts (whose total value is dN), each liquidity endowment of d must on average settle debts with a value of dR. Since dR is the maximum amount of debt that an endowment can settle, every liquidity endowment must settle this amount of debt, and this is precisely what takes place in the case of C_N^1 that I just discussed. I now show that this is impossible for C_N^K when $K > R > 1$. Since each firm owes d/K to K firms, settling debts with a value of dR requires that RK firms make payments of d/K to a creditor.

The first firm i that receives a liquidity endowment has a debt of d outstanding and makes payments of d/K to K firms with indices $i+j$ where j is between 1 and K. If $R > 1$, each of these firms must make further payments so that each endowment distribution leads at least $1+K$ contiguous firms to make payments. If any of these $1+K$ firms receives a subsequent endowment, this endowment cannot be used to make K payments, which implies that fewer than RK firms use this endowment to make payments.

If a later endowment is given to a firm with index $j < i$ such that $j + K > i$, one of the payees of firm j must be firm i. Since firm i has already discharged his obligations, it makes no further payments, so that the endowment given to firm j clears fewer than RK debts of d/K. It follows that a necessary condition for an endowment to clear RK debts is that no firm with an index between $i-K-1$ and i be given an endowment. This logic implies that, for any firm k that receives an endowment, no firm with an index between $k-K-1$ and k can receive an endowment if all endowments are to clear RK obligations. This means no more than one firm out of every $1+K$ can be given an endowment if they are each to pay off debts of Rd. If $K \geq R$, the number of firms whose endowments clear this many debts is thus strictly smaller than N/R so more firms must be given an endowment to clear all debts.

Proof of proposition 7

Each firm i that receives an endowment makes three payments. The first of these goes to $i+1$, who uses it to pay $i+3$ who uses it to pay $i+5$ who uses it to pay $i+8$. The second of these goes to $i+2$, who uses it to pay $i+5$ who uses it to pay $i+6$ who uses it to pay $i+8$. The last goes to $i+3$, who uses it to pay $i+6$ who uses it to pay $i+7$ who uses it to pay $i+8$. Let endowments be distributed to firms 0, 4 and 8. Denote by 1, 2 and 3 the three payments made by 0, by 4, 5 and 6 the three payments made by 4, and by 7, 8 and 9 the three payments made by 8. The table below then shows which payments are received by each firm, except that for firms 1, 4 and 8, it specifies both the payment that they receive (under column R) and that they pay out (under column E). The table demonstrates that each firm j receives payments from $j-1$, $j-2$ and $j-3$ while it makes payments to firms $j+1$, $j+2$, and $j+3$.

	Firm j														
	0		1	2	3	4		5	6	7	8		9	10	11
	R	E				R	E				R	E			
Received from $j-1$	6	1	1	8	9	9	4	4	2	3	3	7	7	5	6
Received from $j-2$	5	2	7	2	1	8	5	1	5	4	2	8	4	8	7
Received from $j-3$	4	3	8	9	3	7	6	2	3	6	1	9	5	6	9

This construction may seem arbitrary but notice that, aside from all ending in $i+8$, the three paths have the property that, in total, they involve four payments each of lengths 1, 2 and 3.

Proof of proposition 8

Suppose that 3 endowments given to firms i, j, and k. If any paths of payments starting at i, j, or k end up at $m \neq i, j, k$, m has at least one unfulfilled obligation (since it neither has an endowment nor is able to use one of the payments it receives to make a further payment). Thus, the paths of payments originating at i, j and k must terminate at i, j, or k for all obligations to be fulfilled. Because $R = 4$ and $K = 2$, the maximum length of a payment chain is 8 and the minimum is 4. Since the path from i to i has length 12, this implies that chains of payments originating in i cannot end at i, so they must end at either j or k.

If the 3 firms receiving endowments are equidistant so that, for example $j = i + 4$ and $k = j + 4$, some obligations are unfulfilled. The reason is that there is then only one path originating in i and terminating in j and the only path from i to k passes through j as well (because the distance between i and k is 8, the path from i to k involves 4 segments of length 2). Thus, equidistant endowments imply either that both of i's endowments end at j (meaning that one only makes two payments) or that j only originates one payment from his own endowment.

This implies that, to fulfill all obligations, one distance between firms receiving endowments, say that between i and j must be smaller than 4. This implies that no payment originating in i terminates in j. Since the distance between j and i is at least 9 (when $j - i = 3$), no payment originating in j ends in i either. Thus, to fulfill all obligations, all payments originating in both i and j have to terminate in k, which is impossible.

Figure 1: An example of vertical obligations

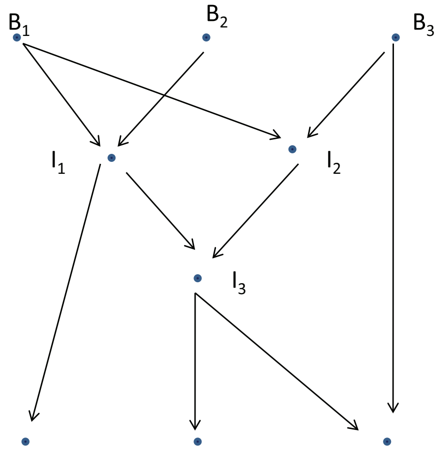

Figure 2: An example combining vertical lending with a cycle

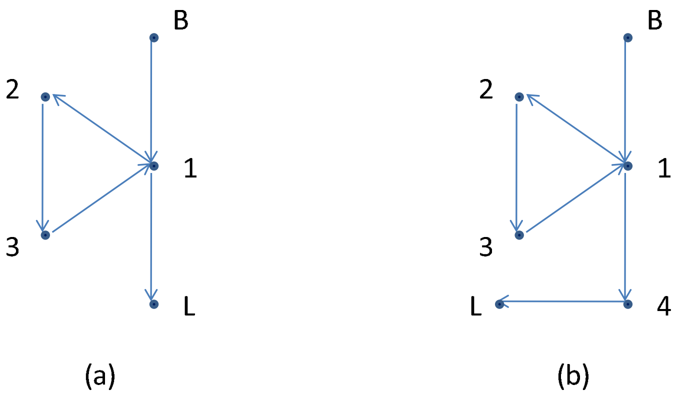

Figure 3: An example with $K = 1$

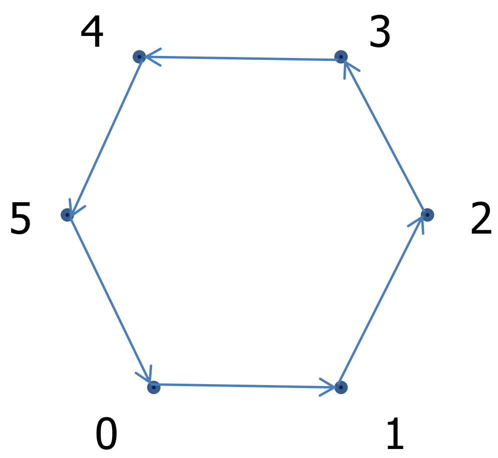

Figure 4: An example with $K = 2$

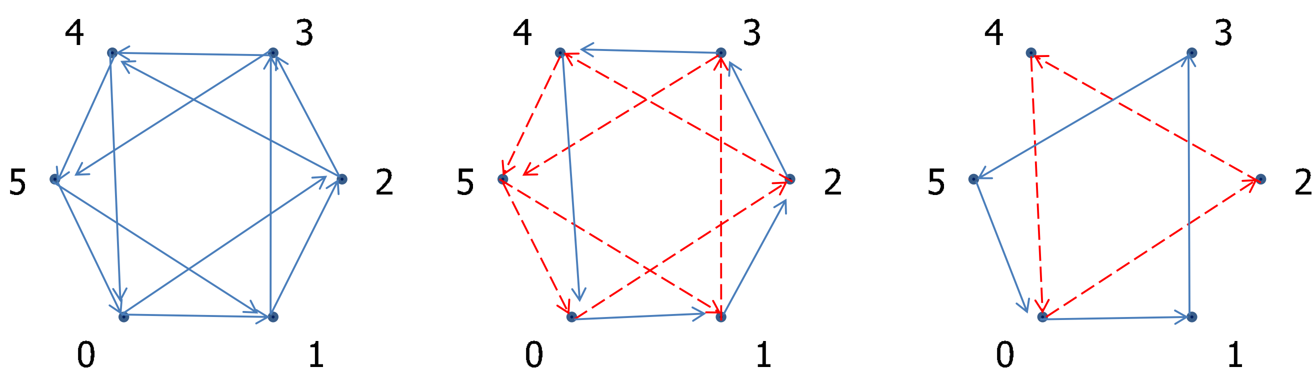

Figure 5: An example with $K = 2$ and a missing cycle

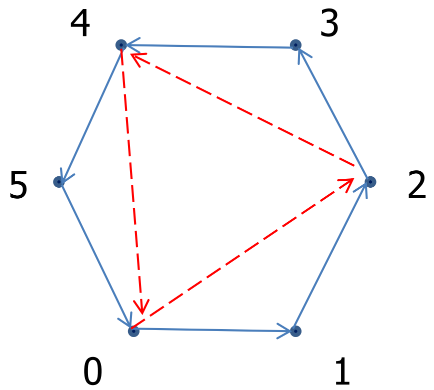

Figure 6: Remaining obligations of C_6^2 when $R = 2$