# Bank liquidity, interbank markets and monetary policy

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**Abstract**

Interbank markets play a vital role for the lending of liquidity among banks with idiosyncratic shocks. This paper examines how efficiently the interbank market distributes liquidity among banks after shocks, and whether this affects banks' choice of liquidity provision to depositors. We show that there are multiple ex-ante Pareto-ranked rational expectations equilibria. There exists a first best equilibrium, in which a low interbank lending rate provides efficient risk-sharing among banks when shocks occur. A high interbank rate is necessary in the state without shocks to induce banks to hold optimal liquidity and provide optimal risk-sharing for their depositors' liquidity needs. The central bank can select the optimal interbank rate equilibrium, in which rates vary according to the state of the financial system, as an optimal monetary policy.

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# 1 Introduction

The appropriate role of a central bank's interest rate policy response to financial disruptions is the subject of continuing debate. A standard view is that monetary policy only plays a role if a financial disruption affects inflation or the real economy. However, central banks appear to often decrease interest rates during disruptions even when output and inflation are not expected to fall, which has led to criticism. For example, in May 2008, Buiter (2008) asked "Despite these worrying inflation developments, and with output not exactly falling off a cliff (and probably not even weakening enough to accommodate the necessary external rebalancing of the US economy) the Fed cut rates aggressively. What accounts for this anomalous, and in my view misguided, monetary policy behaviour?" Goodfriend (2002), discussing earlier episodes, wrote "Consider the fact that the Fed cut interest rates sharply in response to two of the most serious financial crises in recent years: the October 1987 stock market break and the turmoil following the Russian default in 1998. Arguably, in retrospect, interest rate policy remained too easy for too long in both cases."

The framework we develop in this paper suggests that lower interbank market rates during financial disruptions is part of an optimal policy by the central bank. A primary role for banks under incomplete markets is to provide greater risk-sharing and liquidity than markets can provide to depositors who face uninsurable idiosyncratic liquidity shocks. During financial disruptions, which we think of as states when banks face considerable uncertainty regarding their idiosyncratic needs for liquid assets, banks themselves may have large borrowing needs in the interbank market. We show that an interbank market can achieve the optimal allocation–allowing banks to provide efficient risk-sharing to their depositors and insuring banks against their idiosyncratic liquidity shocks–provided the interest rate on this market is state-dependent and low in states of financial disruption. The need for a state dependent interest rate suggests a role for the central bank.

In our model, the interest rate on the interbank market plays two roles: From an ex ante perspective, the expected rate influences the banks' portfolio decision between liquid short-term assets and illiquid long-term assets. Ex post, the rate determines the terms at which banks can trade their assets in response to idiosyncratic shocks. There is a trade-off between the two roles: If the ex ante expected rate is equal to the ex post realized rate, the efficient allocation cannot be achieved. If the rate is low, the redistribution of assets between banks subject to idiosyncratic shocks will be efficient, but banks will choose a suboptimal portfolio. At the rate that induces banks to invest in the optimal portfolio, the interbank market does not achieve an optimal redistribution. If the interbank rate is state dependent, however, the ex ante expected rate need not be equal to the ex post rate in every state. A high expected rate can induce banks to hold the optimal portfolio while a low rate in states of financial disruption allows the efficient redistribution of assets between banks.

There are multiple rational expectations equilibria of our model, only one of which is efficient. The central bank can be thought of as an equilibrium selection device. In particular, we show how the central bank can implement the efficient allocation by setting the interest rate in the interbank market.

Despite the key role they play for financial stability, there is relatively little work studying interbank markets. This may be related to the fact that until recently there was no theory in which interbank markets were part of an optimal arrangement. In their seminal study, Battacharya and Gale (1987) examine banks with idiosyncratic liquidity shocks from a mechanism design perspective. The optimal arrangement in their paper is not an interbank market. More recent work by Freixas and Holthausen (2005), Freixas and Jorge (2008), Heider, Hoerova, and Holthausen (2008), assumes the existence of interbank markets despite the fact they are not part of an optimal arrangement.

Both Allen, Carletti and Gale (2008) and our paper develop frameworks in which interbank markets are efficient. In Allen, Carletti and Gale (2008) the central bank can buy and sell assets, using its balance sheet to achieve the efficient allocation. In contrast, the size of the central bank's balance sheet does not change in our model but the interbank market rate is state dependant. Both of these approaches seem to capture some aspects of actual central bank policy and it may be interesting, in future research, to explore how they may be combined.

Our central bank intervention provides an alternative model to that of Guthrie and Wright (2000) for the concept of "open mouth operations," by which the central bank can determine short term interest rates without active trading intervention in equilibrium. Goodfriend and King (1987) argue that with efficient interbank markets, monetary policy should respond to aggregate but not idiosyncratic liquidity shocks. We find a role for monetary policy to insure banks against shocks to the distribution of liquidity. Results of our paper are similar to Diamond and Rajan (2008) in showing a benefit to reducing interest rates during a crisis, but which leads to moral hazard for bank liquidity holding, and requires a symmetric interest rate policy with high rates in good times. In Diamond and Rajan (2008), providing liquidity to banks through interest rate policy is ineffective and cannot lower interest rates without taxing consumers outside of the banking system because of a Ricardian Equivalence argument. In our framework, the central bank can lower interest rates after distributional shocks to bank liquidity for risk-sharing reasons, because the inelasticity of banks' short term supply and demand for liquidity. Our paper also relates to Bolton et al. (2008) in examining the efficiency of financial intermediaries' choice of holding liquidity versus acquiring liquidity supplied by the market after shocks occur. Efficiency depends on the timing of central bank intervention in Bolton et al. (2008), whereas the level of interest rate policy is the focus of our paper. Ashcraft, McAndrews and Skeie (2008) examine ex-post liquidity trading with credit and participation frictions in the interbank market. The model results explain their empirical findings of reserves hoarding by banks and interbank rate volatility.

# 2 Model

The baseline real model adds distributional bank liquidity shocks and an interbank market to the standard Diamond and Dybvig (1983) framework. There are three dates, denoted by $t = 0, 1, 2$. There is a large number of competitive banks, each with a unit continuum of consumers. Ex-ante identical consumers are endowed with one unit of good at date 0 and learn their private type at date 1. With a probability $\lambda$, a consumer is "early" and needs to consume at date 1, and with complementary probability $1 - \lambda$ a consumer is "late" and needs to consume at date 2.

There are two possible technologies. The short-term liquid technology allows for storing goods at date 0 or date 1 for a return of one the following period. The long-term investment technology allows for investing goods at date 0 for a return of $r > 1$ at date 2. Investment is illiquid and cannot be liquidated at date 1.[2]

Banks are ex-ante identical at date 0. At date 1 learn they their private type

$$j = \{ \begin{array}{ll} a & \text{with prob } \frac{1}{2} \\ b & \text{with prob } \frac{1}{2}, \end{array}$$

with half of banks type $a$ and half type $b$. Bank $j$ has a fraction of early consumers at date 1 equal to

$$\lambda^{j\varepsilon} = \{ \begin{array}{ll} \lambda + \varepsilon & \text{for } j = a \\ \lambda - \varepsilon & \text{for } j = b, \end{array}$$

where $0 < \lambda^{b\varepsilon} \leq \lambda^{\alpha\varepsilon} < 1$, and where $\varepsilon$ is a liquidity shock "state" of the world given by

$$\varepsilon = \{ \begin{array}{ll} \varepsilon' > 0 & \text{with prob } \rho \\ \varepsilon'' = 0 & \text{with prob } 1 - \rho. \end{array}$$

Bank $j$ has a fraction of late consumers at date 2 equal to $1 - \lambda^{j\varepsilon}$.

Consumer utility is

$$U = \{ \begin{array}{ll} u(c_1) & \text{for "early" with prob } \lambda \\ u(c_2) & \text{for "late" with prob } 1 - \lambda, \end{array}$$

where $c_t$ is consumption at date $t = 1, 2$ and $u$ is increasing and concave.

At date 0, consumers deposit their good in their bank for a deposit contract that pays a non-contingent amount for withdrawal at date 1 of $c_1 \geq 0$, or pays an equal share of the bank's remaining goods for withdrawal at date 2 of $c_2^{j\varepsilon} \geq 0$. Consumer's expected utility is

$$E[U] = \lambda u(c_1) + \rho[\tfrac{1}{2}(1 - \lambda^{a\prime})u(c_2^{a\prime}) + \tfrac{1}{2}(1 - \lambda^{b\prime})u(c_2^{b\prime})] + (1 - \rho)(1 - \lambda)u(c_2^{\prime\prime}),$$

where $c_2^{\prime\prime} \equiv c_2^{a\varepsilon^{\prime\prime}} = c_2^{b\varepsilon^{\prime\prime}}$, "$\prime$" denotes "$\varepsilon^\prime$" and "$\prime\prime$" denotes "$\varepsilon^{\prime\prime}$." Banks maximize their depositors expected utility and make zero profit because of competition for deposits at date 0. Banks invest $\alpha \in [0, 1]$ in long-term assets and store $1 - \alpha$ in liquid goods. At date 1, consumers and banks learn their private type. Bank $j$ borrows $f^{j\varepsilon} \in \mathbb{R}$ on the interbank market and consumers withdraw. At date 2, bank $j$ repays the amount $f^{j\varepsilon} l^{\varepsilon}$ for its loan and the bank's remaining consumers withdraw. We assume, for now, that the interbank lending gross rate of return $l^{\varepsilon} \geq 1$. We check later that $l^{\varepsilon} < 1$ cannot occur in equilibrium, since storage is available.

[2] We extend the model to allow for liquidation at date 1 in Section 5.

The budget constraints for bank $j$ in state $\varepsilon$ for dates 1 and 2 are

$$
\begin{aligned}
\lambda^{j\varepsilon} c_1 &= 1-\alpha-\beta^{j\varepsilon}+f^{j\varepsilon} \qquad \text{for } j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\} \qquad (1)\\
(1-\lambda^{j\varepsilon})c_2^{j\varepsilon} &= \alpha r+\beta^{j\varepsilon}-f^{j\varepsilon} l^{\varepsilon} \qquad \text{for } j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}, \qquad (2)
\end{aligned}
$$

respectively, where $\beta^{j\varepsilon} \in [0, 1-\alpha]$ is the amount of liquid goods that bank $j$ stores between dates 1 and 2. We assume that banks lend goods when indifferent between lending and storing. We also assume that banks cannot contract with each other at date 0 and that $c_1$ is non-contingent. Late consumers bear all the risk of liquidity shocks in $c_2^{j\varepsilon}$.[3] Further, we assume that the coefficient of relative risk aversion for $u(c)$ is greater than one, which implies that banks provide risk-decreasing liquidity insurance. Throughout the paper, we disregard sunspot-triggered banks runs. For now, we consider parameters such that there are no bank defaults in equilibrium.[4] As such, we assume that incentive compatibility holds:

$$c_2^{j\varepsilon} \geq c_1 \text{ for all } j \in \{a,b\} \text{ and for } \varepsilon \in \{\varepsilon', \varepsilon''\}.$$

This rules out both standard multiple equilibria bank runs as well as bank runs based on very large $\varepsilon'$ shocks.

From the date 1 budget constraint (1), we can solve for

$$f^{j\varepsilon} = \lambda^{j\varepsilon} c_1^j - (1-\alpha) + \beta^{j\varepsilon}.$$

Substituting this in the date 2 budget constraint (2) and rearranging gives

$$c_2^{j\varepsilon} = \frac{\alpha r + \beta^{j\varepsilon} - [\lambda^{j\varepsilon} c_1 - (1-\alpha) + \beta^{j\varepsilon}] l^{\varepsilon}}{(1-\lambda^{j\varepsilon})}. \qquad (3)$$

A bank's optimization to maximize its depositors' expected utility is

$$
\begin{aligned}
\max_{\alpha\in[0,1];c_1,\{\beta^{j\varepsilon}\}_{j,\varepsilon}\geq 0} \quad & E[U] && (4)\\
\text{s.t.} \quad & \beta^{j\varepsilon} \leq 1-\alpha \qquad \text{for } j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\} && (5)\\
& (3) \qquad \text{for } j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}, && (6)
\end{aligned}
$$

where the constraint gives the maximum amount of goods that can be stored between dates 1 and 2.

[3] Allowing $c_1$ to depend on $\varepsilon$ or $j$ would complicate the solutions but would not change the qualitative results of the model. Note, in particular, that in the optimal allocation, $c_1$ is constant.

[4] Bank runs are considered in section **??**

## 3 Results

To find the first best allocation, consider a planner who can observe consumer types. The planner can consider the aggregate economy, for which there are no aggregate shocks, with no need to consider bank types $a$ and $b$. The planner's problem is to maximize a consumer's expected utility

$$\begin{aligned} \max_{\alpha\in[0,1];c_1,\beta\geq 0} \quad & \lambda u(c_1) + (1-\lambda)\, u(c_2) \\ \text{s.t.} \quad & \lambda c_1 \leq 1 - \alpha + \beta \\ & (1-\lambda)\, c_2 \leq \alpha r + 1 - \alpha - \beta - \lambda c_1 \\ & \beta \leq 1 - \alpha, . \end{aligned}$$

The constraints are the physical quantities of goods available for consumption at date 1 and 2, and available storage between dates 1 and 2, respectively. The first-order conditions and binding constraints give the well-known first best allocations, which are denoted with asterisks, defined by

$$u'(c_1^*) = ru'(c_2^*) \tag{7}$$

$$\lambda c_1^* = 1 - \alpha^* \tag{8}$$

$$(1-\lambda)\, c_2^* = \alpha^* r. \tag{9}$$

$$\beta^* = 0 \tag{10}$$

Equation (7) shows that the ratio of marginal utilities between dates 1 and 2 is equal to the marginal return on investment $r$.

We next consider bank $j's$ optimization (4).

**Lemma 1.** *First order conditions with respect to $c_1$ and $\alpha$ are, respectively,*

$$u'(c_1) = E[\frac{\lambda^{j\varepsilon}}{\lambda} l^{\varepsilon} u'(c_2^{j\varepsilon})] \tag{11}$$

$$E[l^{\varepsilon} u'(c_2^{j\varepsilon})] = rE[u'(c_2^{j\varepsilon})]. \tag{12}$$

**Proof.** The Lagrange multiplier for constraint (5) is $\theta_{\beta}^{j\varepsilon}$. The first order condition with respect to $\beta^{j\varepsilon}$ is

$$\tfrac{1}{2}\rho^{\varepsilon} u'(c_2^{j\varepsilon})(1-l^{\varepsilon}) \leq \theta^{j\varepsilon} \qquad (= \text{ if } \beta^{j\varepsilon} > 0),$$

which for $l^\varepsilon > 1$ does not bind and implies $\beta^{j\varepsilon} = 0$, and for $l^\varepsilon = 1$ implies $\beta^{j\varepsilon} = 0$ since banks are indifferent between storage and lending goods. Complementary slackness for constraint (5) implies $\theta_\beta^{j\varepsilon} = 0$. First order conditions (11) and (12) follow. ■

Equation (11) is the Euler equation and determines the investment level $\alpha$ given $l^\varepsilon$. Equation (12) states that the expected marginal utility-weighted returns on storage and investment must be equal. The return on investment between dates 0 and 2 is $r$. The return on storage between dates 0 and 2 is the market rate $l^\varepsilon$. Banks can store goods at date 0, lend them at date 1, and will receive $l^\varepsilon$ at date 2. The rates $l'$ and $l''$ are determined in equilibrium to make banks indifferent to holding goods and assets at date 0.

The interbank market clearing condition is

$$f^{a\varepsilon} = -f^{b\varepsilon} \text{ for } \varepsilon \in \{\varepsilon', \varepsilon''\},$$

which with the bank's budget constraints (1) and (2) determine $c_1^j$ and $f^{j\varepsilon}$ as functions of $\alpha$:

$$\begin{aligned} c_1(\alpha) &= \frac{1-\alpha}{\lambda} \\ f^{j\varepsilon}(\alpha) &= (1-\alpha)(\frac{\lambda^{j\varepsilon}}{\lambda} - 1). \end{aligned}$$

Finding the market equilibrium is reduced to solving the two first order conditions (11) and (12) in three unknowns: $\alpha$, $l'$ and $l''$.

### 3.1 Single state: $\rho = 0, 1$

We start by finding solutions to the special cases of $\rho = 0, 1$. We then apply these solutions to solve the general model $\rho \in [0, 1]$ below. There is certainty about the single state of the world $\varepsilon$ at date 1. First order conditions (11) and (12) can be written more explicitly as

$$\begin{aligned} & \rho[\tfrac{1}{2}u'(c_2^{a\prime}) + \tfrac{1}{2}u'(c_2^{b\prime})]\boldsymbol{l}' + (1-\rho)u'(c_2'')\boldsymbol{l}'' \\ = \; & \rho[\tfrac{1}{2}u'(c_2^{a\prime}) + \tfrac{1}{2}u'(c_2^{b\prime})]\boldsymbol{r} + (1-\rho)u'(c_2'')\boldsymbol{r} \end{aligned} \tag{13}$$

$$u'(c_1) = \rho[\tfrac{\lambda^{a\prime}}{2\lambda}u'(c_2^{a\prime}) + \tfrac{\lambda^{b\prime}}{2\lambda}u'(c_2^{b\prime})]l' + (1-\rho)u'(c_2'')l'' \tag{14}$$

Equations (13) and (14) imply that for $\rho = 0$, the value of $l'$ is indeterminate, and for $\rho = 1$, the value of $l''$ is indeterminate. In either case, we will show that there is an equilibrium with unique values for the allocation $c_1$, $c_2$ and $\alpha$. The indeterminate variable is of no consequence for the allocation. The allocation is determined by the two first order equations, in the two unknowns $\alpha$ and $l''$ (for $\rho = 0$) or $l'$ (for $\rho = 1$). The first order condition with respect to $\alpha$, equation (13), shows that the interbank lending rate equals the return on assets: $l^{''} = r$ (for $\rho = 0$) or $l' = r$ (for $\rho = 1$). With a single state of the world, the interbank lending rate must equal the return on assets.

In the case of no shock with $\rho = 0$, the banks' budget constraints imply that in equilibrium $f^{a\prime\prime} = f^{b\prime\prime} = 0$, no interbank lending occurs. The interbank lending rate $l''$ is the lending rate at which banks net borrowing demand is zero. The Euler equation for banks equation (14) is equivalent to that for the planner equation (7). Banks choose the optimal $\alpha^*$ and provide the first best allocation $c_1^*$ and $c_2^*$, which are illustrated in Figure 1a.

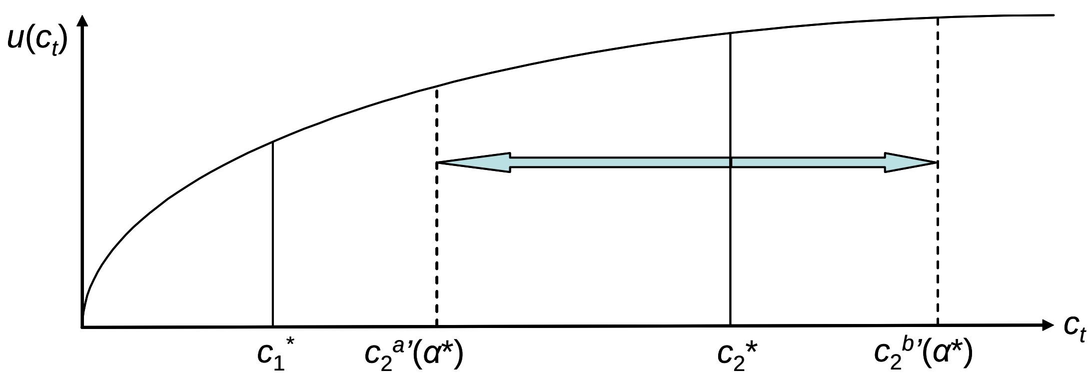


Figure 1a

Banks provide liquidity at date 1 to early consumers by paying $c_1^* > 1$. This can only be accomplished by paying $c_2^* < r$ on withdrawals to late consumers at date 2. The key for the bank being able to provide liquidity insurance to early consumers is that the bank can only pay an implicit date 1 to date 2 intertemporal return on deposits of $\frac{c_2^*}{c_1^*}$, which is less than the return on assets $r$. This contract is optimal because the ratio of intertemporal marginal utility equals the marginal return on assets, $\frac{u'(c_2^*)}{u'(c_1^*)} = r$.

**Proposition 1.** *For $\rho = 0$, there exists a rational expectations equilibrium characterized by $l' = r$ that has a unique first best allocation $c_1^*$, $c_2^*$, $\alpha^*$.*

**Proof.** For $\rho = 0$, equation (13) implies $l'' = r$. Equation (14) simplifies to $u'(c_1) = u'(c_2'')r$, and the bank's budget constraints bind and simplify to $c_1 = \frac{1-\alpha}{\lambda}$, $c_2 = \frac{\alpha r}{1-\lambda}$. These results are equivalent to the planner's results in equations (7) through (9), implying there is a unique equilibrium, where $c_1 = c_1^*$, $c_2 = c_2^*$ and $\alpha = \alpha^*$. ■

In the case of a certain shock with $\rho = 1$, there is interbank lending. The banks' budget constraints imply that in equilibrium $f^{a\prime} = \varepsilon' c_1$ and $f^{b\prime} = -\varepsilon' c_1$. First, consider the outcome at date 1 holding fixed $\alpha = \alpha^*$. With $l' = r$, late consumers do not have optimal consumption: $c_2^{a\prime}(\alpha^*) < c_2^* < c_2^{b\prime}(\alpha^*)$. The deviation from optimality is illustrated by the arrows in Figure 1a. A bank that has to borrow at date 1 at the rate $l' = r$ faces a rate that is higher than the intertemporal return on deposits $\frac{c_2}{c_1}$ and cannot pay late consumers

$$c_2' \equiv \frac{\alpha r}{1 - \lambda}.$$

Late consumers face risk to their consumption conditional on being a late type. Second, consider the determination of $\alpha$. In equilibrium, $\alpha > \alpha^*$. Compared to the first best, banks store fewer liquid goods at date 0 and pay lower $c_1$ at date 1 in order to hold more assets that provide banks greater self-insurance liquidity available at date 2 to pay to late consumers. The difference of equilibrium consumption written as a function of the equilibrium $\alpha$ compared to consumption for a fixed $\alpha = \alpha^*$ is demonstrated by the arrows in Figure 1b. The result is $c_1 < c_1^*$, $c_2' > c_2^*$, $c_2^{a\prime} > c_2^{a\prime}(\alpha^*)$ and $c_2^{b\prime} > c^{b\prime}(\alpha^*)$. For any $\varepsilon > 0$ shock, banks do not provide the optimal allocation.

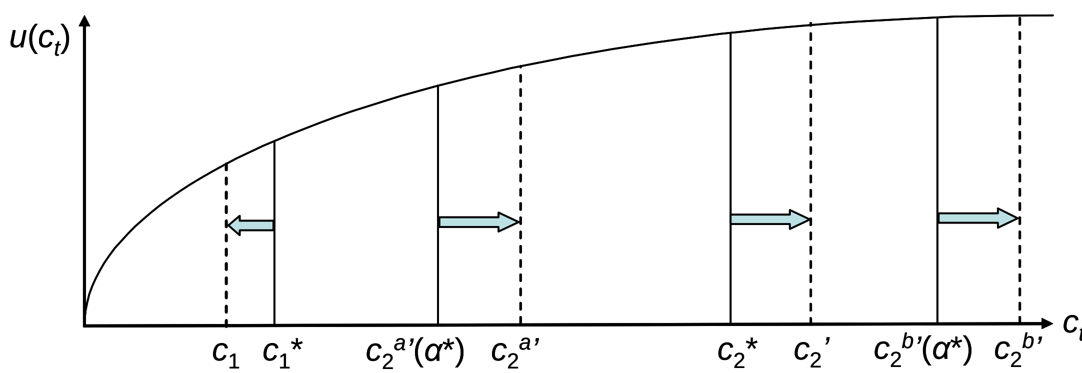


Figure 1b

**Proposition 2.** *For $\rho = 1$, there exists a rational expectations equilibrium characterized* *by $l' = r$ that has a unique suboptimal allocation*

$$\begin{aligned} c_1 &< c_1^* \\ c_2^{a\prime} &< c_2^* < c_2^{b\prime} \\ \alpha &> \alpha^*. \end{aligned}$$

**Proof.** For $\rho = 1$, equation (13) implies $l' = r$. By equation (3), $c_2^{b\prime} > c_2^{a\prime}$. From the bank's budget constraints and market clearing,

$$\frac{1-\lambda-\varepsilon}{2(1-\lambda)}c_2^{a\prime} + \frac{1-\lambda+\varepsilon}{2(1-\lambda)}c_2^{b\prime} = \frac{\alpha r}{1-\lambda} = c_2',$$

which implies $\frac{1}{2}c_2^{a\prime} + \frac{1}{2}c_2^{b\prime} < c_2'$, since $c^{b\prime} > c_2^{a\prime}$. Because $u(\cdot)$ is concave, $\frac{1}{2}u'(c_2^{a\prime}) + \frac{1}{2}u'(c_2^{b\prime}) > u'(c_2')$. Further, $\frac{\lambda^{a\prime}}{2\lambda}u'(c_2^{a\prime}) + \frac{\lambda^{b\prime}}{2\lambda}u'(c_2^{b\prime}) > u'(c_2')$ since $\lambda^{a\prime} > \lambda^{b\prime}$, $\frac{\lambda^{a\prime}}{2\lambda} + \frac{\lambda^{b\prime}}{2\lambda} = 1$ and $c_2^{a\prime} < c_2^{b\prime}$. Thus,

$$\begin{aligned} u'(c_1(\alpha^*)) &= ru'(c_2'(\alpha^*)) \\ &< r[\frac{\lambda^{a\prime}}{2\lambda}u'(c_2^{a\prime}(\alpha^*)) + \frac{\lambda^{b\prime}}{2\lambda}u'(c_2^{b\prime}(\alpha^*))]. \end{aligned}$$

Since $u'(c_1(\alpha))$ is increasing in $\alpha$ and $u'(c_2^{j\prime}(\alpha))$ for $j = a, b$ is decreasing in $\alpha$, the Euler equation implies that in equilibrium, $\alpha > \alpha^*$. Hence, $c_1 = \frac{1-\alpha}{\lambda} < c_1^*$, $c_2^{b\prime} > c_2' = \frac{\alpha r}{1-\lambda} > c_2^*$ and $c_2^{a\prime} < c_2^*$. ■

### 3.2 General shock: $\rho \in [0, 1]$

We now apply our results of the special cases of $\rho = 0, 1$ to examine the general case of $\rho \in [0, 1]$. We will show that there are multiple rational expectations equilibria with different real allocations of $c_1$, $c_2$ and $\alpha$.

There are two possible states of the world at date 1: $\varepsilon'$ and $\varepsilon''$. An equilibrium is determined by the two first order condition equations (13) and (14) in three unknowns $\alpha$, $l'$ and $l''$. The bank's budget constraints imply in the state of no shock with $\varepsilon'' = 0$, no interbank lending occurs, $f^{ja} = f^{jb} = 0$, and

$$c_2'' = \frac{\alpha r}{1-\lambda},$$

as in the case of $\rho = 0$. In the state of a positive shock with $\varepsilon' > 0$, there is interbank lending with $f^{a\prime} = \varepsilon' c_1$, $f^{b\prime} = -\varepsilon' c_1$,

$$c_2^{j\varepsilon} = \frac{\alpha r - (\lambda^{j\varepsilon} - \lambda)c_1 l^{\varepsilon}}{1-\lambda^{j\varepsilon}}. \tag{15}$$

In particular, there exists a suboptimal rational expectations equilibrium with $l' = l'' = r$. Consider $l' = r$. Equation (13) implies $l'' = r$. Equation (14) is a single equation with a single unknown $\alpha$, which is determined. Equation (14) implies that $\alpha(\rho)$ is an implicit function of $\rho$. Likewise, $c_2''(\rho)$, $c_2^{a\prime}(\rho)$ and $c_2^{b\prime}(\rho)$ are implicit functions of $\rho$. We can use the cases of $\rho = 0$ and $\rho = 1$ to provide bounds for the general case of $\rho \in [0, 1]$. The equilibrium $c_1(\rho)$ and $c_2^{j\varepsilon}(\rho)$ for $j = a, b$ and $\varepsilon = \varepsilon', \varepsilon''$, written as functions of $\rho$, are displayed in Figure 2a. This figure shows that $c_1(\rho)$ is decreasing in $\rho$ while $c_2^{j\varepsilon}(\rho)$ in increasing in $\rho$. In particular,

$$\begin{aligned} c_2^* = c_2''(0) &\leq c_2''(\rho) \leq c_2''(1) \\ c_2^{j\varepsilon}(0) &\leq c_2^{j\varepsilon}(\rho) \leq c_2^{j\varepsilon}(1) \\ c_1(1) &\leq c_1(\rho) \leq c_1(0) = c_1^*, \end{aligned}$$

for $j = a, b$, where $c_2^{j\prime}(\rho = 0) = c_2^{j\prime}(\alpha = \alpha^*)$. With interbank rates equal to $r$ in all states, there is inefficient risk-sharing among late consumers. To compensate, there is inefficient liquidity provided to early consumers.

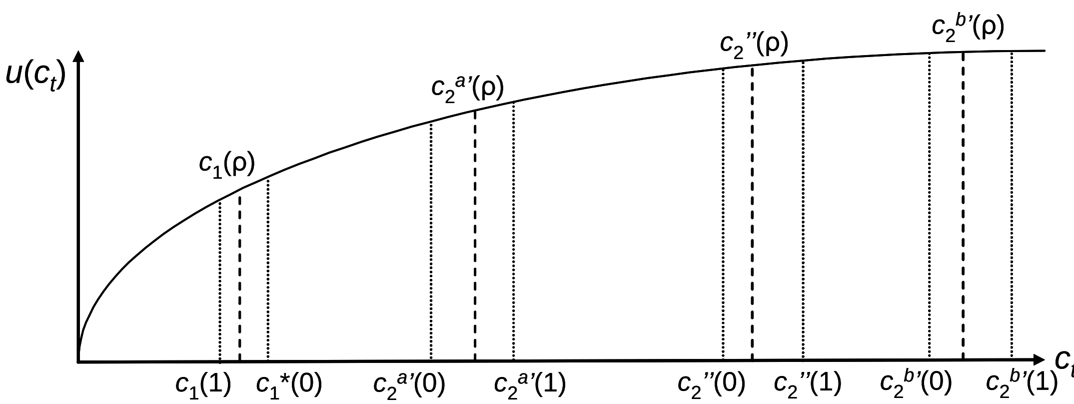


Figure 2a

For $\rho < 1$, there also exists a first best rational expectations equilibrium with

$$l' = \bar{l}' \equiv \frac{c_2''}{c_1}. \tag{16}$$

To show this, first we substitute for $l'$ into (15). and simplify, which gives $c_2^{a\prime} = c_2^{b\prime} = c_2' = c_2'' = \frac{\alpha r}{1-\lambda}$. With $l'$ equal to the intertemporal return on deposits between dates 1 and 2, there is optimal ex-post risk-sharing of the goods that are available at date 2 through interbank lending at the low rate at date 1. Substituting for $l'$ and $c_2^{j'}$ into equation (13) and rearranging gives

$$l'' = r + \frac{\rho(r - \frac{c_2}{c_1})}{1-\rho}. \tag{17}$$

Substituting for $c_2^{j'}$, $l'$ and $l''$ into equation (14) and rearranging gives $u'(c_1) = r'u'(c_2')$. This is the planner's condition, and implies $\alpha = \alpha^*$, $c_1 = c_1^*$ and $c_2' = c_2^*$, a first best allocation. To interpret, substituting these equilibrium values into (17) and simplifying shows that

$$l'' = \bar{l}'' \equiv r + \frac{\rho(r - \frac{c_2^*}{c_1^*})}{1-\rho} > r, \tag{18}$$

whereas $l' = \frac{c_2^*}{c_1^*} < r$. With $l''$ greater than $r$ during the no-shock state, there is no ex-post inefficiency because there is no need for interbank lending. With $l'$ less than $r$ for the shock state, there is no ex-post inefficiency with interbank lending because the rate is at the low optimal rate. The following result shows that the expected interbank rate is equal to the return on assets. This result is based the first order condition with respect to $\alpha$, which requires banks to be willing to hold both storage and investment at date 0.

**Proposition 3.** *The expected interbank rate is* $E[l^\varepsilon] = r$.

**Proof.** $E[l^\varepsilon] = \rho l' + (1-\rho)l''$. Substituting for $l'$ and $l''$ from (16) and (18) and simplifying, $E[l^\varepsilon] = r$. ■

Since there is no risk to late consumers, banks hold optimal $\alpha^*$. Figure 2b illustrates the distinction of this first best equilibrium with $l' = \frac{c_2^*}{c_1^*}$, $\bar{l}''$ from the equilibrium with $l' = l'' = r$. Arrows indicate that in contrast with the suboptimal $l^\varepsilon = r$ equilibrium, in the $l' = \frac{c_2^*}{c_1^*}$ equilibrium we find the first best outcome that $c_2^{j\varepsilon}(\rho) = c_2^*$ and $c_1(\rho) = c_1^*$ for all $j \in \{\alpha, b\}$, $\varepsilon \in \{\varepsilon', \varepsilon''\}$ and $\rho < 1$.

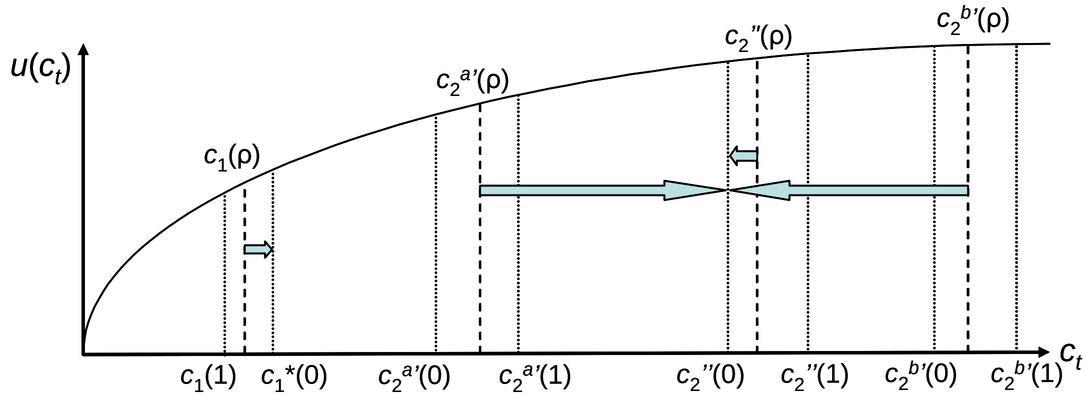

Figure 2b

For $\rho = 1,\ l' = \frac{c_2''}{c_1}$ would imply $l''$ is not finite and equations (13) and (14) are not well specified. Therefore we rule out $l' = \frac{c_2''}{c_1}$ as an equilibrium value for $\rho = 1$. As in the case of $\rho = 1$ above, there are multiple equilibria since $l'$ is indeterminate, but the allocation $\alpha$, $c_1$, $c_2^{j''}$ is unique and not first best.

**Proposition 4.** *For $\rho \in (0, 1)$, there exist multiple rational expectations equilibria with different allocations. There exists a suboptimal rational expectations equilibrium with*

$$\begin{aligned} l' &= l'' = r \\ \alpha &> \alpha^* \\ c_1 &< c_1^* \\ c_2'' &> c_2^* \\ c_2^{a\prime} &< c_2^* < c_2^{b\prime}, \end{aligned}$$

*and there exists a first best rational expectations equilibrium with*

$$\begin{aligned} l' &= \frac{c_2^*}{c_1^*} < r \\ l'' &= \bar{l}'' > r \\ \alpha &= \alpha^* \\ c_1 &= c_1^* \\ c_2^{j\varepsilon} &= c_2^* \text{ (for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}\text{)}. \end{aligned}$$

Note that all equilibria we consider are rational expectations equilibria. Allen and Gale (2004) show that there exist sunspot equilibria in this type of model. From the perspective of date 1 only, an indeterminate continuum of $l^\varepsilon$ is consistent with ex-post individual rationality for banks lending in the interbank market. We show that multiple rational expectations equilibria exist from the perspective of date 0 because there are multiple $\varepsilon$ idiosyncratic liquidity states at date 1. There is a family of $l', l''$ at date 1, each pair of which can be anticipated and support a different rational expectations equilibrium. Within a rational expectations equilibrium, $l'$ and $l''$ do not need to be equal. The results from this section generalize in a straightforward way to the case of $N$ shocks, as shown in appendix B.

### 3.3 Central bank monetary policy

The interest rate $l^\varepsilon$ at which banks lend in the interbank market is equivalent to the unsecured interest rate that many central banks target for monetary policy, such as with the fed funds rate targeted by the Federal Reserve in the U.S. The role of the central bank for monetary policy in the model is to select the optimal interbank rate equilibrium among multiple equilibria for $\rho \in (0, 1)$. The central bank setting the interbank rate at a low rate $l' = \frac{c_2^*}{c_1^*}$ after the shock $\varepsilon'$ is equivalent to a transfer from bank $b$ to bank $a$. An equilibrium with $l' = \frac{c_2^*}{c_1^*}$ has ex-post distributional effects as bank $b$ late consumers receive less and bank $a$ late consumers receive more than in the equilibrium with $l' = r$, but the $l' = \frac{c_2^*}{c_1^*}$ is ex-ante optimal because it reduces risk for banks' late consumers. Banks then do not need to self-insure for the $\varepsilon'$ shock with greater investment, and so will hold greater liquidity for its early consumers of $1 - \alpha = 1 - \alpha^*$. But extra high rates of $l' = \bar{l}'' > r$ are required after the no-shock state $\varepsilon''$, such that expected rates equal the return on assets, $E[l^\varepsilon] = r$, and banks are indifferent between holding goods and assets at date 0.

The model we have used so far is not rich enough to adequately describe how the central bank can implement the desired interbank interest rate. The main benefit of this simple model it to illustrate the key point of the paper without the burden of too much notation. However, the role of the central bank is central to our argument and we provide a generalized version of the model that shows how the central bank can actively select and enforce its choice of interbank rates in appendix A. In that richer model, banks can pay a fiat nominal rate rather than a real rate on deposits, following Skeie (2008). The central bank can offer to borrow and lend unlimited amounts at its nominal policy rate contingent on the state $\varepsilon$ at date 1. This will force banks to trade at this rate in the interbank market, and the central bank does zero borrowing and lending in equilibrium. The equilibrium of the nominal model is equivalent in real terms to the equilibrium of the real model in Section 3.2.

## 4 Bank runs

We extend the model to consider bank runs, and we show that banks runs may occur if the CB does not follow the optimal policy. In the state where $\varepsilon > 0$, patient depositors of banks with many impatient agents will consume less than patient depositors of other banks if the CB does not set the interest rate equal to $c_2/c_1$, the intertemporal return on deposits. If $\varepsilon$ is sufficiently large it may be the case that the consumption of patient depositors of banks with many impatient agents would be lower than the consumption of impatient depositors, which would trigger a run.

This argument can be presented in several ways. One can find the equilibrium allocation assuming that the CB does not follow the optimal policy and show that, in equilibrium, banks runs would occur at banks that have many impatient depositors. Alteratively, one can consider an equilibrium assuming that the CB follows the optimal policy and show that, if the CB makes an unexpected mistake, a bank run occurs. We consider each approach in turn.

## 4.1 Central bank makes unexpected mistake

If the CB is assumed to adopt the optimal policy, the equilibrium allocation is optimal. Suppose that, unexpectedly, the CB chooses interest rate $l' = r > c_2^*/c_1^*$ in the state where $\varepsilon = \varepsilon' > 0$. In this case, the consumption, $c_2^{a'}$, of patient depositor in banks with many impatient agents is

$$c_2^{a'} = \frac{\alpha^* r - \varepsilon' c_1^* r}{1-\lambda-\varepsilon'} = \frac{r}{1-\lambda-\varepsilon'}\left[\alpha^* - \varepsilon' \frac{1-\alpha^*}{\lambda}\right],$$

since $c_1^* = (1-\alpha^*)/\lambda$. If we assume that the utility function is of the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 1,$$

then we can rewrite the expression for $c_2^{a'}$ as

$$c_2^{a'} = \frac{\alpha^* r}{1-\lambda}\left[\frac{1-\lambda-\varepsilon' r^{\frac{\sigma-1}{\sigma}}}{1-\lambda-\varepsilon'}\right].$$

Recall that $\varepsilon' \leq \min\{\lambda, 1-\lambda\}$. If $\lambda$ is very small, then $\varepsilon'$ must also be very small and the term in brackets will be close to 1. This implies that $c_2^{a'}$ will be close to $c_2^*$ and no bank run can occur since $c_2^* > c_1^*$. In contrast, if $\lambda \geq 1/2$, then the term in brackets can be made arbitrarily close to zero, since $r > 1$ so that $c_2^{a'}$ will be close to zero. In such cases, bank runs can occur.

Consider the following example: $\lambda = 1 - \lambda = 1/2$, $r = 1.5$, and $\sigma = 2$. For such parameters, we have $\alpha^* \approx 0.4495$, $c_1^* \approx 1.101$, and $c_2^* \approx 1.3485$. Now assume that $\varepsilon' = 0.3$, then $c_2^{a'} \approx 0.8939 < c_1^*$, so there would be a bank run.

### 4.2 Runs in equilibrium

Consider the equilibrium allocation if banks anticipate that the interbank market interest rate will be $l' = l'' = r$. By continuity, this allocation converges to the optimal allocation as $\rho \to 0$. We have already seen that at the optimal level $\alpha^*$, bank runs can occur if $\varepsilon$ is sufficiently large and $l' = r$. Now since bank runs are anticipated, banks could choose a "run preventing" deposit contract, as suggested by Cooper and Ross (1998). However, following the argument in that paper, banks will not choose a run-preventing deposit contract if the probability of a bank run is sufficiently small. So for $\rho$ sufficiently close to zero, bank runs will occur in equilibrium.

## 5 Liquidation of the long-term technology

We extend the model to allow for liquidation of the investment at date 1. We show that this restricts possible real interbank rates and may preclude the first best equilibrium. At date 1, bank $j$ liquidates $\gamma^{j\varepsilon}$ of the investment for a salvage rate of return $s$ at date 1 and no further return at date 2. The bank budget constraints (1) and (2) are replaced by

$$\begin{aligned} \lambda^{j\varepsilon} c_1 &= 1 - \alpha - \beta^{j\varepsilon} + \gamma^{j\varepsilon} s + f^{j\varepsilon} \qquad \text{for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\} \\ (1 - \lambda^{j\varepsilon}) c_2^{j\varepsilon} &= (\alpha - \gamma^{j\varepsilon}) r + \beta^{j\varepsilon} - f^{j\varepsilon} l^{\varepsilon} \qquad \text{for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}, \end{aligned}$$

and the bank optimization (4) is replaced

$$\begin{aligned} \max_{\alpha, c_1, \{\beta^{j\varepsilon}, \gamma^{j\varepsilon}\}_{j,\varepsilon}} \quad & E[U] \\ \text{s.t.} \quad & \beta^{j\varepsilon} \leq 1 - \alpha \qquad \text{for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\} \\ & \gamma^{j\varepsilon} \leq \alpha \qquad \text{for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}. \end{aligned} \tag{19}$$

The first order condition with respect to $\gamma^{j\varepsilon}$ is

$$\tfrac{1}{2} \rho^{\varepsilon} u'(c_2^{j\varepsilon})(l^{\varepsilon} s - r) \leq \theta_{\gamma}^{j\varepsilon} \quad \text{for } j \in \{a, b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\} \qquad (= \text{ if } \gamma^{j\varepsilon} > 0), \tag{20}$$

where $\theta_\gamma^{j\varepsilon}$ is the Lagrange multiplier for constraint (19). Without loss of generality, we assume that no bank $j$ liquidates all investment in state $\varepsilon$ unless all banks do. Because the interbank market is ex-post efficient, the equilibrium and allocation depends solely on the aggregate amount of liquidation, not the distribution of liquidation among banks. If there is complete liquidation of investment, then clearly the allocation is not first best.

Consider an equilibrium in which there is not complete liquidation of investment. Complementary slackness for constraint (19) implies $\theta_\gamma^{j\varepsilon} = 0$. Condition (20) can be written as

$$l^\varepsilon \leq \frac{r}{s} \qquad \text{for all } \varepsilon,$$

which gives a restriction on the equilibrium interest rate in state $\varepsilon$. If there is liquidation by any bank $j$ in any state $\varepsilon$, the equilibrium is not first best. Alternatively, if $\bar{l}'' > \frac{r}{s}$, then the equilibrium cannot be first best. The interest cannot be high enough in the $\varepsilon''$ state. At an interest rate of $l'' > \frac{r}{s}$, all banks would liquidate investment and lend it on the interbank market, and no banks would borrow, which cannot be an equilibrium.

## 6 Conclusion

This paper examines the ex-ante choice of bank liquidity and the ex-post reallocation of bank liquidity through the interbank market after random idiosyncratic liquidity shocks. An expected high rate equal to the return on long-term assets is required for banks to hold liquidity ex-ante. In the state when the liquidity shock occurs, banks in need borrow from banks with surplus funds. A high interbank rate, however, is inefficient for lending in the interbank market. The return on long-term assets is necessarily greater than the implicit return that banks pay on deposits to late consumers. This implicit rate is low to allow banks to provide their key role of liquidity insurance to early consumers. Banks that borrow on the interbank market at the high rate pay their late consumers a lower rate on deposits than that paid by banks that lend on the interbank market. The uncertainty of returns to late consumers implies that banks hold less liquidity as insurance to consumers against individual idiosyncratic liquidity shocks in order to provide more liquidity as insurance for late consumers against bank idiosyncratic liquidity shocks.

However, there are multiple rational expectations equilibria, which are Pareto-ranked on an ex-ante basis. An alternative equilibrium can reach the first best allocation. An implicit interbank rate during the no-shock state that is greater than the return on assets allows for a lower interbank rate during the shock state. The return on liquid goods in expectation, equal to the expected interbank rate, equals the return on assets and makes banks indifferent between holding liquid goods and assets. The low interbank rate during the shock state allows for ex-post efficient interbank lending, such that late consumers receive equal rates from banks that have positive or negative shocks. Banks do not have to provide extra liquidity at the late period to self-insure against bank shocks. Banks are able to hold optimal liquid goods for liquidity insurance against consumer shocks.

The interest rate policy of a central bank can select the interbank rate equilibrium and allocation to consumers. According to the model, a central bank should lower interest rates after an idiosyncratic liquidity shock. This has a distributional effect that benefits banks with negative liquidity shocks and costs banks with positive liquidity shocks, but which is ex-ante optimal for banks before they learn their shock. In order to lower rates after a shock and still ensure banks maintain incentives to hold liquidity, a central bank must raise rates above the "natural" return on long-term assets during the no-shock state. Rates should be raised in a symmetric manner to how they are lowered in the different states, adjusting for the probability of the shock occurring. Examining the impact of monetary policy on bank liquidity and the interbank market when there are aggregate liquidity shocks and real shocks to fundamental asset values are steps for future research.

# 7 Appendix A: Monetary policy with nominal rates

We expand the real model to allow for nominal interbank lending rates. With nominal fiat interest rates, the central bank can explicitly enforce its target for the interbank rate, in order to actively select the rational expectations equilibrium. The central bank offers to borrow and lend to banks any amount of nominal, fiat money at the central bank's policy rate at date 1, which ensures that the interbank market rate equals the central bank's policy rate. The equilibrium and allocation of the nominal rate model is equivalent to the real rate model.

## 7.1 Nominal rate model extension

The extension of the model to include nominal rates is based on Skeie (2008). A nominal unit of account, inside money and a goods market with firms are added to the model of banks with real deposits. To establish a fiat nominal unit of account, the central bank offers at date 0 to buy or sell goods to the extent feasible for fiat currency at a fixed price $P_0 = 1$. After date 0, the central bank does not set the price of goods and does not offer to buy or sell goods. At date 0, each bank makes a loan to a firm. The firm buys the good from the bank's unit continuum of consumers, and consumers deposit in the bank. All the transactions at date zero are paid for in the amount of one nominal unit of account. These nominal payments are called "inside money." The individual budget constraint for each of the banks, firms and consumers requires that the net inside money payments of each party nets to zero at date 0.

Each bank lends to its firm for loan repayments of nominal amounts $(1 - \alpha) K_1$ and $\alpha K_2^{\varepsilon}$ payable in inside money at dates 1 and 2, respectively. Uppercase variables denote nominal values and lowercase variables denote real values. The firm buys the good from consumers for price $P_0 = 1$. The firm invests $\alpha$ and stores $1 - \alpha$ of the good, where $\alpha$ is chosen and can be enforced by the bank.[5] Consumers deposit in their bank for a demand deposit contract repayment due in inside money of either $D_1 \geq 1$ if withdrawn at date 1 or $D_2^{j\varepsilon} \geq 1$ if withdrawn at date 2. Although no currency circulates, the central bank's

[5] If the bank could not enforce the firm's storage, the bank could alternatively buy and store $1 - \alpha$ goods, sell them at date, and lend $\alpha$ to the firm without any storage requirements. Results of the model would be unchanged.

offer to trade with currency establishes the nominal unit of account for transactions with bank inside money. This is equivalent to Skeie (2008), where currency rather than inside money transactions occur at date 0.

At each date $t = 1, 2$, payments are made among banks with either inside money or currency. Each bank's net payments in a period must equal zero. At date 1, $\lambda^{j\varepsilon}$ early consumers of bank $j$ withdraw to buy goods from a firm in the goods market. At date 2, $1-\lambda^{j}$ late consumers withdraw from banks to buy goods. The representative firm repays loans and banks borrow or lend if needed on the interbank market or from the central bank.

The bank's budget constraints from the real model (1) and (2) are replaced by budget constraints for nominal payments:

$$\text{s.t.} \qquad \lambda^{j\varepsilon} D_1 = (1-\alpha)K_1 + M_f^{j\varepsilon D} + M_h^{j\varepsilon D}, \qquad \forall\ j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}, \tag{21}$$

$$(1-\lambda^{j\varepsilon})D_2^{j\varepsilon} = \alpha K_2^{\varepsilon} - M_f^{j\varepsilon D} D_f^{\varepsilon} - M_h^{j\varepsilon D} D_h^{\varepsilon}, \qquad \forall\ j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon', \varepsilon''\}, \tag{22}$$

respectively, where bank $j$'s demand to borrow from other banks is $M_f^{\varepsilon j D}$ and from the central bank (in currency) is $M_h^{\varepsilon j D}$, and where $D_f^{\varepsilon}$ and $D_h^{\varepsilon}$ are the returns on interbank loans and central bank loans, respectively. $D_f^{\varepsilon}$ is the interbank market rate, which is determined in equilibrium. At date 1, the central bank targets $D_f^{\varepsilon}$ by choosing its policy rate $D_h^{\varepsilon}$ at which it offers to borrow and lend to banks an unlimited amount. Specifically, the central offers to lend $M_h^{j\varepsilon S}(D_h^{\varepsilon}) \in (-\infty, \infty)$ to bank $j$ at rate $D_h^{\varepsilon}$, where $M_h^{S}(D_h^{\varepsilon})$ is a correspondence. The central does not have a budget constraint to equate its borrowing and lending of central bank money. It can create and destroy money as needed. The central bank's lending supply is perfectly elastic at its chosen rate $D_h^{\varepsilon}$. The way in which we model the central bank offering to borrow and lend at a single policy rate is similar to open market operations in practice. Many central banks in essence offer to borrow and lend an elastic amount of funds at a chosen rate to target the interbank rate at which banks lend uncollateralized to each other. Open market operations lending is often collateralized in practice, as in the form of repos against government securities in the case of the Federal Reserve. We abstract from collateralization since there is no risk of loss or default.

Consumers buy goods from firms at date $t = 1, 2$ in a Walrasian market using inside money as numeraire. Consumption for early and late consumers is

$$c_1(P) = \frac{D_1}{P_1} \tag{23}$$

$$c_2^{j\varepsilon}(P) = \frac{D_2^{j\varepsilon}}{P_2^{\varepsilon}}, \tag{24}$$

where $P_t^{\varepsilon}$ is the price of goods at date $t = 1, 2$ and $P \equiv (P_1, P_2^{\varepsilon})$ is a vector. We consider only $P_t^{\varepsilon} \in (0, \infty)$, which is for simplicity and does not effect the results. Consumers' aggregate demand is given by

$$q_1^D(P) = \frac{\frac{1}{2}(\lambda^{a\varepsilon} + \lambda^{b\varepsilon})D_1}{P_1} \tag{25}$$

$$q_2^D(P) = \frac{\frac{1}{2}[(1-\lambda^{a\varepsilon}) + (1-\lambda^{b\varepsilon})]D_2^{j\varepsilon}}{P_2^{\varepsilon}}. \tag{26}$$

The representative firm submits a supply schedule $q_t^{\varepsilon S}(P_t^{\varepsilon})$ for the goods market. The firm's optimization is to maximize profits:

$$\max_{q_1^{\varepsilon S}, q_2^{\varepsilon S} \geq 0} \quad 1 - \alpha + \alpha r - q_1^{\varepsilon S} - q_2^{\varepsilon S} \tag{27a}$$

$$\text{s.t.} \quad q_1^{\varepsilon S} \leq 1 - \alpha \tag{27b}$$

$$q_2^{\varepsilon S} \leq 1 - \alpha + \alpha r - q_1^{\varepsilon S} \tag{27c}$$

$$q_1^{\varepsilon S} \geq \frac{(1-\alpha)K_1}{P_1} \tag{27d}$$

$$q_2^{\varepsilon S} \geq \frac{\alpha K_2^{\varepsilon}}{P_2^{\varepsilon}}. \tag{27e}$$

The objective function (27a) is the profit in goods that the firm consumes at date 2. Constraints (27b) and (27c) are the maximum amounts of goods that can be sold at dates 1 and 2, respectively. Constraints (27d) and (27e) are the firm's budget constraints to repay its loan at date 1 and date 2, respectively.

The bank's demand for borrowing on the interbank market can be solved for from equation (21) as

$$M_f^{\varepsilon j D} = \lambda^{\varepsilon j} D_1 - (1-\alpha)K_1 - M_h^{j\varepsilon D}. \tag{28}$$

Substituting for $M_f^{\varepsilon j D}$ from equation (28) into equation (22) and rearranging, we find that bank $j$ pays withdrawals to late consumers the amount

$$D_2^{j\varepsilon} = \frac{\alpha K_2^{\varepsilon} - [\lambda^{j\varepsilon} D_1 - (1-\alpha)K_1]D_f^{\varepsilon} + (D_f^{\varepsilon} - D_h^{\varepsilon})M_h^{j\varepsilon D}}{1 - \lambda^{j\varepsilon}}. \tag{29}$$

The bank's optimization problem (4) is replaced by

$$\max_{\alpha\in[0,1];D_1\geq 0;\{M_h^{j\varepsilon D}\}_{j,\varepsilon}\in\mathbb{R}} \quad E[U], \tag{30}$$

$$\text{s.t.} \quad (29) \tag{31}$$

where $c_1(P)$ and $c_2^{j\varepsilon}(P)$ are given by (23) and (24), respectively.

An *equilibrium* is defined as goods market prices and quantities $(P, q_1, q_2)$, deposit and loan returns and quantities $\{D_1, D_f^\varepsilon, M_h^{j\varepsilon}\}_{j,\varepsilon}$, and investment $(\alpha)$ that solve goods market clearing conditions

$$q_t^D(P) = \tfrac{1}{2}[q_t^{aS}(P) + q_t^{bS}(P)] \text{ for } t = 1, 2,$$

and interbank market clearing condition

$$M_f^{aD}(M_h^{aD}) + M_f^{bD}(M_h^{bD}) = 0, \tag{33}$$

where $\{\alpha, D_1, M_h^{j\varepsilon}\}_{j,\varepsilon}$ is a solution to bank $j$'s optimization (30); $\{q_t^D(P)\}_{t=1,2}$ is given by the consumers' aggregate demand (25) and (26), and $(q_1^{\varepsilon S}(P), q_2^{\varepsilon S}(P))$ is a solution to the firm's optimization (27).

### 7.2 Nominal rate results

The results of the nominal model are equivalent to those of the real model, with the addition that the central bank can choose its policy rate to target the interbank rate. The first order conditions for bank $j'$s optimization (30) with respect to $\alpha$, $c_1$ and $M_h^{\varepsilon jD}$are

$$\begin{aligned}
E[\frac{K_2^\varepsilon}{P_2^\varepsilon}u'(c_2^{j\varepsilon})] &= E[\frac{K_1}{P_2^\varepsilon}D_f^\varepsilon u'(c_2^{j\varepsilon})], \qquad \forall\, \varepsilon \in \{\varepsilon', \varepsilon''\} &(34)\\
E[\frac{1}{P_1}u'(c_1)] &= E[\frac{\lambda^{j\varepsilon}D_f^\varepsilon}{\lambda P_2^\varepsilon}u'(c_2^{j\varepsilon})], \qquad \forall\, \varepsilon \in \{\varepsilon', \varepsilon''\} &(35)\\
D_f^\varepsilon &= D_h^\varepsilon, \qquad \forall\, \varepsilon \in \{\varepsilon', \varepsilon''\}, &(36)
\end{aligned}$$

respectively. Loan returns are set according to a competitive loan market as

$$\begin{aligned}
K_1 &= P_1 &(37)\\
K_2^\varepsilon &= rP_2^\varepsilon, &(38)
\end{aligned}$$

such that the real returns $\frac{K_1}{P_1} = 1$ and $\frac{K_2^\varepsilon}{P_2^\varepsilon} = r$ equal the marginal product of capital for their respective terms and firms make zero profits in equilibrium. Substituting for $K_t^\varepsilon$

from equations (37) and (38), conditions (34) and (35) can be written as

$$E[u'(c_2^{j\varepsilon})] = E[\frac{D_f^\varepsilon}{P_2^\varepsilon/P_1}u'(c_2^{j\varepsilon})] \tag{39}$$

$$E[u'(c_1)] = E[\frac{\lambda^{j\varepsilon}}{\lambda}\frac{D_f^\varepsilon}{P_2^\varepsilon/P_1}u'(c_2^{j\varepsilon})]. \tag{40}$$

Condition (36) states that because of arbitrage, the interbank rate $D_f^\varepsilon$ equals the central bank's policy rate $D_h^\varepsilon$. The real interbank rate equals the nominal rate divided by nominal goods price inflation between dates 1 and 2:

$$l^\varepsilon = \frac{D_f^\varepsilon}{P_2^\varepsilon/P_1}, \tag{41}$$

which implies that the first order conditions for the nominal model, equations (39) and (40), and for the real model, equations (13) and (14), are equivalent. The central bank can target any real interbank lending rate $l^\varepsilon$ at date 1 by choosing

$$D_h^\varepsilon = \frac{P_2^\varepsilon}{P_1}l^\varepsilon,$$

subject to satisfying the date 0 first order conditions for $l'$ and $l''$. In particular, the central bank can implement the first best allocation by choosing

$$D_h^\varepsilon = \overline{D}_h^\varepsilon \equiv \frac{P_2^\varepsilon}{P_1}\overline{l}^\varepsilon. \tag{42}$$

**Proposition 5.** *The central bank can choose $D_h^\varepsilon = \overline{D}_h^\varepsilon$, and there exists a unique equilibrium with first best allocation $\alpha = \alpha^*$, $c_1 = c_1^*$ and $c_2^{j\varepsilon} = c_2^*$.*

**Proof.** Equilibrium prices and quantities satisfy

$$P_1 = \frac{\lambda D_1}{q_1} \tag{43}$$

$$P_2^\varepsilon = \frac{(1-\lambda)D_2}{q_2}. \tag{44}$$

The constraints in the firm's optimization (27) bind, which gives

$$q_1 = 1-\alpha \tag{45}$$

$$q_2 = \alpha r. \tag{46}$$

Substitution for quantities and prices from (43) - (46) into (23) and (24),

$$c_1 = \frac{1-\alpha}{\lambda} \tag{47}$$

$$c_2 = \frac{\alpha r}{1-\lambda}. \tag{48}$$

To find $D_1$, substituting for $M_f^{j\varepsilon D}$ from (28) into the market clearing condition (33) and simplifying gives

$$D_1 = \frac{(1-\alpha)\,K_1}{\lambda} + \frac{M_h^{a\varepsilon D} + M_h^{b\varepsilon D}}{2\lambda}. \tag{49}$$

Substituting from (49) for $D_1$ into (28) and simplifying gives the demand for interbank borrowing by bank $j$ as

$$M_f^{j\varepsilon D} = (\frac{\lambda^{j\varepsilon}}{\lambda} - 1)\,(1-\alpha)\,K_1 + \frac{\lambda^{j\varepsilon}}{\lambda}(M_h^{a\varepsilon D} + M_h^{b\varepsilon D}) - M_h^{j\varepsilon D}.$$

Rearranging, aggregate bank borrowing is

$$M_f^{j\varepsilon D} + M_h^{j\varepsilon D} = (1-\alpha)\,K_1(\frac{\lambda^{j\varepsilon}}{\lambda} - 1) + \frac{\lambda^{j\varepsilon}}{\lambda}(M_h^{a\varepsilon D} + M_h^{b\varepsilon D}), \tag{50}$$

Using (50), we can show that

$$(M_f^{a\varepsilon D} + M_f^{b\varepsilon D}) + (M_h^{a\varepsilon D} + M_h^{b\varepsilon D}) = 2(M_h^{a\varepsilon D} + M_h^{b\varepsilon D}). \tag{51}$$

By market clearing equation (33), aggregate net interbank borrowing is zero, $M_f^{a\varepsilon D} + M_f^{b\varepsilon D} = 0$, which by equation (50) implies $(M_h^{a\varepsilon D} + M_h^{b\varepsilon D}) = 2(M_h^{a\varepsilon D} + M_h^{b\varepsilon D})$. Hence, $(M_h^{a\varepsilon D} + M_h^{b\varepsilon D}) = 0$. Aggregate net borrowing from the central bank is zero in equilibrium. The central bank lends zero net supply of liquidity to the market. While bank $j$ aggregate net borrowing from the interbank market and the central bank is determined by equation (50) as $M_f^{j\varepsilon D} + M_h^{j\varepsilon D} = (1-\alpha)\,K_1(\frac{\lambda^{j\varepsilon}}{\lambda} - 1)$, the individual components $M_f^{j\varepsilon D}$ and $M_h^{j\varepsilon D}$ are not determined. The central bank does not need to lend to any banks in equilibrium. Lending by the central bank is equivalent and a substitute for interbank lending.

Substition into (29) for $K_t$ from (37) and (38), for $D_h^{\varepsilon}$ from (36), for $D_h^{\varepsilon}$ from (42), for $1-\alpha = \lambda c_1$ from (47), and for $D_1 = \frac{P_1 q_1}{\lambda} = \frac{P_1(1-\alpha)}{\lambda}$ from (43) and (45), and rearranging gives

$$c_2^{j\varepsilon} = \frac{D_2^{j\varepsilon}}{P_2^{\varepsilon}} = \frac{\alpha r - (\lambda^{j\varepsilon} - \lambda)c_1\bar{l}^{\varepsilon}}{1-\lambda^{j\varepsilon}},$$

which is identical to $c_2^{j\varepsilon}$ in the real model given by equation (15). The bank has an optimization identical to that in the real model and chooses $\alpha = \alpha^*$. Hence, the equilibrium is identical to that of the real model and the allocation is $c_1 = c_1^*$ and $c_2^{j\varepsilon} = c_2^*$. ■

## 8 Appendix B: Generalization to N shocks

Consider the case of the baseline real model (without the central bank, nominal rates, runs or liquidation of assets) where $\varepsilon$ can take N values, $\varepsilon_1, ..., \varepsilon_N \geq 0$. We maintain the assumption that $\varepsilon_1 = 0$. The probability of $\varepsilon_i$ is $\rho_i$, $\sum_{i=1}^{N} \rho_i = 1$.

A bank's problem is thus

$$\begin{aligned}
\max_{\alpha\in[0,1],c_1\geq 0} \quad & \lambda u(c_1) + \sum_{i=1}^{N} \rho_i [\tfrac{1}{2}(1-\lambda^{a\varepsilon_i})u(c_2^{a\varepsilon_i}) + \tfrac{1}{2}(1-\lambda^{b\varepsilon_i})u(c_2^{b\varepsilon_i})] \\
\text{s.t.} \quad & \lambda^{j\varepsilon} c_1 \leq 1 - \alpha + \beta^{j\varepsilon} + f^{j\varepsilon} \\
& (1-\lambda^{j\varepsilon})c_2^{j\varepsilon} \leq \alpha r - \beta^{j\varepsilon} - f^{j\varepsilon} l^{\varepsilon} \\
& \text{for } j \in \{a,b\} \text{ and } \varepsilon \in \{\varepsilon_1, ..., \varepsilon_N\}.
\end{aligned}$$

The first order conditions with respect to $\alpha$ and $c_1$ are, respectively,

$$\sum_{i=1}^{N} \rho_i [\tfrac{1}{2}u'(c_2^{a\varepsilon_i}) + \tfrac{1}{2}u'(c_2^{b\varepsilon_i})] \boldsymbol{l}^{\varepsilon_i} = \sum_{i=1}^{N} \rho_i [\tfrac{1}{2}u'(c_2^{a\prime}) + \tfrac{1}{2}u'(c_2^{b\prime})] \boldsymbol{r} \tag{52}$$

$$u'(c_1) = \sum_{i=1}^{N} \rho_i [\tfrac{\lambda^{a\varepsilon_i}}{2\lambda} u'(c_2^{a\varepsilon_i}) + \tfrac{\lambda^{b\varepsilon_i}}{2\lambda} u'(c_2^{b\varepsilon_i})] l^{\varepsilon_i} \tag{53}$$

By the same logic as in the case with two shocks, the interest rate in the interbank market should be equal to $c_2^*/c_1^*$ whenever $\varepsilon_i > 0$ in order to facilitate risk sharing between banks. Without loss of generality, assume that $\varepsilon_i$ for all $i \geq 2$. Then we have $l^{\varepsilon_i} = c_2^*/c_1^*$ and $c_2^{a\varepsilon_i} = c_2^{b\varepsilon_i} = \frac{\alpha r}{1-\lambda}$ for all $i \geq 2$. Let $\sum_{i=2}^{N} \varepsilon_i = \rho$, then we can write interest rate $l^{\varepsilon_1}$ as

$$l^{\varepsilon_1} = r + \frac{\rho(r - \frac{c_2}{c_1})}{1-\rho}, \tag{54}$$

which is the same as in the two shock case.[6]

[6] We can show that if there is no state with a zero-size shock, then a first best equilibrium does not exist because an equilibrium requires an interest rate of $l^{\varepsilon} > \frac{c_2^*}{c_1^*}$ for at least one state $\varepsilon$, which is then always distortionary. If the baseline real model is modified such that $\varepsilon' > \varepsilon'' > 0$, we can show that there is a constrained efficient equilbrium with $\bar{l}' < l' < r < l'' < \bar{l}''$, which is chosen by the central bank.

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